MATH 4113

Abstract Algebra II

Fall 2024

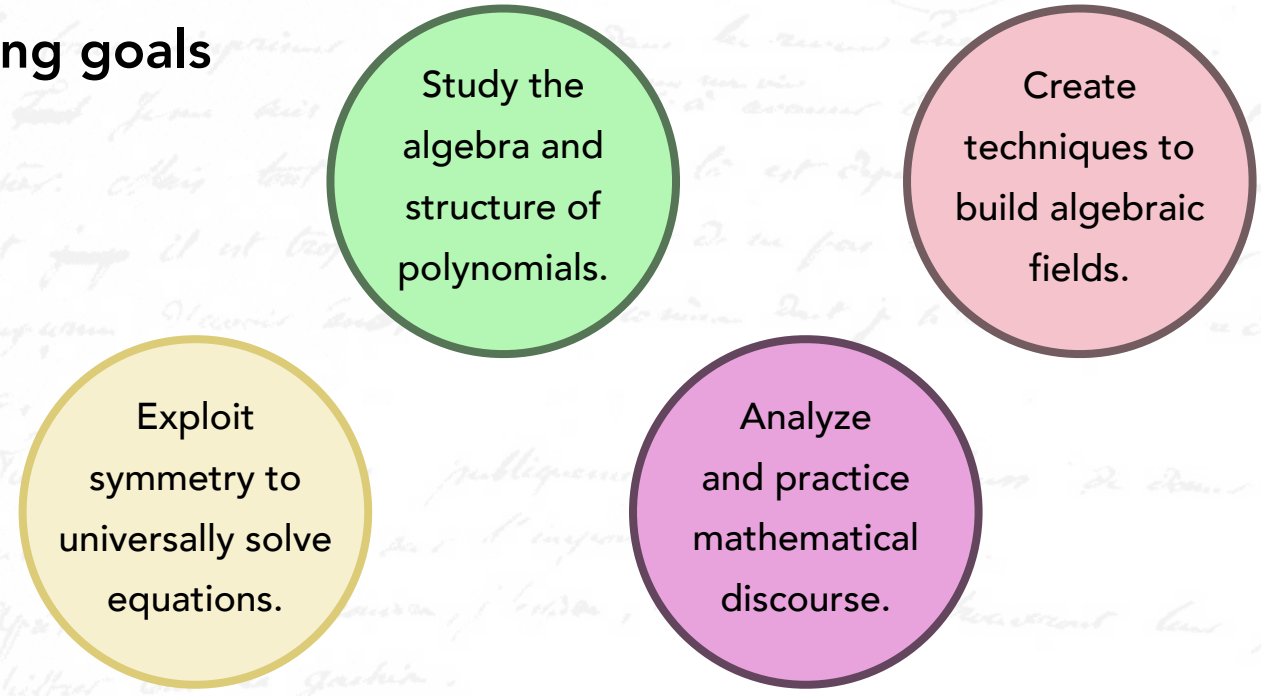
Professor John Bergdall
bergdall@uark.edu

MWF 12:55 - 1:40 PM
Classroom: SCEN 405

Overview

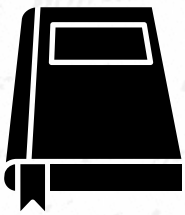
This is a second course in abstract algebra. We will develop the fundamental algebraic laws that govern polynomial equations. We do this by creating connections between algebraic numbers and symmetry groups in algebra.

Learning goals



Page background: handwritten page of manuscript by Évariste Galois, 1832. (Public domain.)

Basic resources

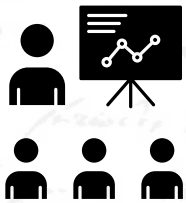


Textbook

Abstract Algebra (4th ed.)
by Beachy and Blair
(Rent/buy: $47/$71*)

Course materials

Materials posted to Blackboard.
Assignments on Gradescope.
(Free)



Office hours (SCEN 220)

Arranged by Bookings MWF
8:30 - 11:30 + MW 2:00 - 3:30.
(See Blackboard for booking info.)

*Used costs. If costs cause hardship or would limit your course access, email me for support.

Tasks and grades

Course tasks are divided into *Supporting Tasks*, *Performance Tasks*, and *Major Works*. Supporting Tasks keep the course moving. Performance Tasks build up your understanding. Major works let you develop crucial theoretical results. **Further details on pages 3-6.**

Supporting Tasks

Reading Quizzes and In-class Presentations.

Performance Tasks

Homework Questions going beyond your reading.

Major works

Four investigations of key theoretical principles.

Turn-in

Reading Quizzes are typically due *Friday* afternoons (5:00 PM). **Performance Tasks** are typically due *Tuesday* afternoons (5:00 PM).

Revisions

Revising your work reinforces your learning. Each task may be revised *once*.

- Revisions for Reading Quizzes are due **Tuesday of the following week** (5:00 PM).
- Revisions for Performance Tasks or Major Works will be collected **on a regular basis** (TBD).

Late work

Completing tasks by their due dates keeps you on track. Nevertheless, you *may turn work in late*.

- Unanswered Reading Quizzes may be turned in through the revision process.
- Performance Tasks may be turned in by **Wednesday** (10:00 AM). Otherwise, they are revisions.
- Your benefit from turning in *any progress* by the initial due dates!
- *Additional* extensions are possible and handled on a case-by-case basis. Contact me early to discuss.

Icon credits on page 8.

Supporting Tasks

Reading Quizzes measure understanding of readings *and* related class discussions. **In-class Presentations** are low-stakes chances to impact the learning environment. **Revisions** of either support your growth as learners.

Reading Quizzes (1 point per question)

- **2-3 questions** per week indicated *ahead of time.*
- **Initial responses** entered in timed quiz and auto-graded on Gradescope.
- **Revisions** are to include complete reasoning, in writing.

Details. Reading Quiz questions are *multiple choice* or *select all*. I will provide prompts. If an initial attempt is not right, revision is possible! Your revision, in writing, should identify the correct response(s) *and* provide a logical explanation for the correct response(s). Half-points are rewarded partial, but incomplete or inaccurate, explanations. Here is what constitutes logical explanation.

- *Multiple choice.* A logical explanation is a proof or justification of the correct response.
- *Select all.* Provide both a positive justification of the correct responses *and* a negative justification of the incorrect ones. (See pg. 9 for an example revision.)

In-class Presentations (1 point per Task)

- **Class welcomes.** Work with a group to write a brief summary of the prior meeting's key terms and ideas. (Rotating basis.)
- **Group presentations.** Work with a group to answer a prompt and explain your reasoning to the class. (Rotating basis.)
- **Revisions** are written re-dos posted for all to benefit.

Details. In-class Presentations sound more intense than they will be! These are *everyday* activities. You will average 1-2 presentations of a few minutes per week. Each Task earns a point.

- *Class welcomes.* A group will write a 4-5 sentence summary of a class meeting and then present it to open the next class. You may be asked to revise your summary, based on feedback from me.
- *Group presentations.* During class time we will work in small groups and the outcomes of your discussions will be presented during the *next* meeting.will start with discussion during one class meeting and your outcome, or conclusion, will be presented at the *next* meeting.

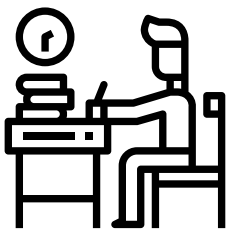
Presenting yourself can be stressful. Students will have access to a variety of office hours, to help smooth out any bumps, prior to class meetings.

Supporting Task Grading

Your Supporting Task grade is the total points earned divided by the total point available. So, 3/4 on quizzes and 2 completed presentations gives 5/6 total for your Supporting Task grade.

Performance Tasks

Performance Tasks are a form of Homework Questions, with the goal of substantially building on specific, important, topics.



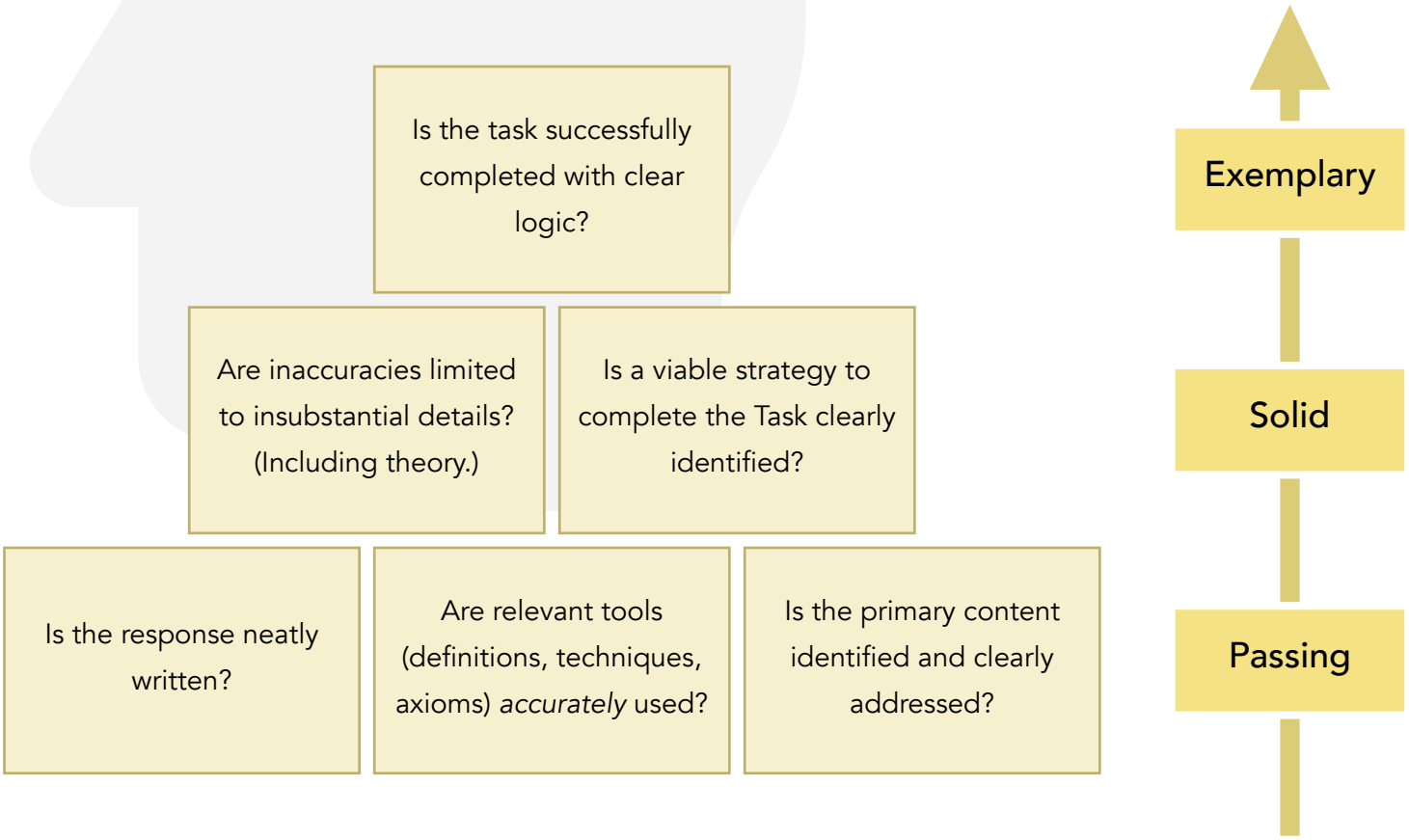
Homework Questions

- **2-3 Questions** due *most* weeks on *Tuesdays.*
- **Collaborate!** As long as you attribute credit to your source(s).
- **Revisions** will be collected regularly.

Details. Performance Tasks are responses to short prompts partially based on class discussion and readings. Each prompt will assign a theoretical or computation problem chosen to build upon something we have already modeled for you, or otherwise discussed. What you turn in can be seen as an initial attempt, since since you will be allowed to revise your work. It is to your benefit to turn in your ideas to get feedback, rather than wait to perfect them alone.

Performance Task Grading

Each Performance Task is graded out of six (6) points, based on qualities of *passing, solid,* and *exemplary* work. See pg. 9-10 for example applications of this rubric.



Major works

Major works let you piece together parts of the theory and develop a narrative explaining a single piece of mathematics.

Major works

- **Essay/paper format.** Typically 5-10 pages, two weeks notice for each.
- **Develop** basic understanding of fundamental laws in algebra.
- **Revisions** will be collected.

Details. The idea of a Major Work is to put the pen to the paper and develop a completely self-contained explanation or illustration of a mathematical idea. In Abstract Algebra 2, we based these works on *four fundamental laws of polynomial algebra*. (I made this terminology up.)

Fundamental laws of polynomial algebra over a field™

- Polynomial equations have no more roots than their degree.
- An algebraic number's degree determines the dimension of the field it generates.
- Every polynomial equation has a *universal* smallest field containing its roots. ("**Splitting fields**")
- The algebraic structure of a splitting field is faithfully reproduced by its symmetry group. ("**Galois theory**")

These laws may have no meaning to you, right now. That is okay! One goal of the course is to build from the smallest law on roots of polynomials to the largest one, known as "Galois theory". Essays related to each law will be assigned slowly throughout the semester. (There is *no* final exam, but rather your final essay on Galois theory will be the final major work.)

Major Work Grading

Your Major Works are graded out of nine (9) points, based on qualities of *passing*, *solid*, and *exemplary* work. The first six points are the same as the Performance Tasks (p. 4). The final three are based on qualities that only make sense in an essay/paper format.

Passing	Solid	Exemplary
Is the narrative self-contained and coherent, with intro and conclusion, focused on content?	Does the essay develop both explanations *and* examples to support the discussion?	Does the essay raise *at least* one original and substantial point beyond the prompt?

Final grade

Your final grade is determined by thresholds in each course component. We *may* make more *generous* thresholds later in the semester.

	To earn a C…	To earn a B…	To earn an A…
Supporting Tasks	at least 70%	at least 80%	at least 90%
Performance Tasks	at least 50%	at least 70%	at least 90%
Major Works	at least 50%	at least 70%	at least 90%

Example. A B-grade means at least 80% in Supporting Tasks *and* 70% in each of Performance Tasks and Major Works. C-level grades are *passing*, B-level grades are for *solid work*, and A-level grades are for *exemplary work*. For an A, you are required at least 90% in *every* category. In terms of rubrics:

- *Performance Tasks*. A-level work is 5's and 6's, while B-level work is 4's and 5's. C-level work is 3's.
- *Major* Works. A-level is an average above 8/9, B-level work is an average above 7/9 and C-level work is something like 5/9 on average.

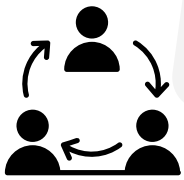
Tips for success

Post! To Blackboard.

Ask! Me early and often.

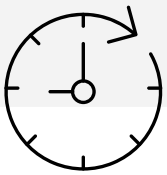
Tips for the stuck and frustrated.

- Take a break. Walk away.
- Explain to someone why you are stuck.
- Check hypotheses or assumptions.
- Work out a single example.
- Search the internet for key terms.
- Keep going!



Collaborate! With peers.

Read! The text is a guide.

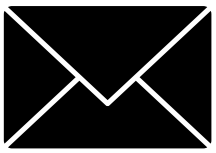


Budget! Take your time.

Further Resources

Appointments

Office hours will not always fit your schedule. Email me to set appointments. In your initial message, please explain (i) what you want to talk about, (ii) *your* availability and (iii) if meeting in person is important (Zoom is possible).



Email

Your questions are important. Email me or post to Blackboard. (I may ask you to do this anyways.) Responses may take 24 hours, but send a reminder if I do not reply.

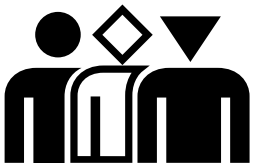
Access

I want you to pass this class, with success. If you find resources not being provided, tell me and I will work with you to fix the issue. For some students, the Center for Educational Access (cea.uark.edu) can coordinate student accommodation requests. Please request your accommodation letter early semester and meet me to discuss it.

Class formation

Names and pronouns

You deserve to be called how you want. Let me know your preferred name and pronouns any time. When meeting one another, re-introduce yourself *frequently.* Everyone's names, pronouns, etc. should be known and properly used.



Diversity

I pledge attention to your identity and experience, regardless of age, background, beliefs, ethnicity, gender identity and expression, national origin, racial identity, religious beliefs, sexual orientation, and any other visible or non-visible categories. You are expected to do the same for myself and your peers.

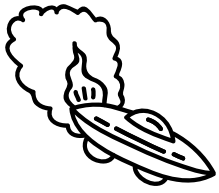
Academic integrity

I trust my students to follow (i) the University of Arkansas Academic Integrity Policy and (ii) my own explicit assignment instructions. When you have questions about a boundary or rule, ask me rather than assuming you know the answer.

Technical policies

Inclement weather

If the University closed due to inclement weather, our course *will not* meet, online or otherwise. Alternative learning materials and assessments will be supplied, to make up the missing class days.



Class cancellation (emergencies)

If class is cancelled on short notice, I will work with the math office to try to find a substitute. If a substitute is found, class will be held. If not, alternative learning materials and assessments will be supplied. Regardless, an announcement will be sent promptly via email and posted to Blackboard.

Schedule

Study the algebra and structure of polynomials.

Create techniques to build algebraic fields.

Exploit symmetry to universally solve equations.

Unit	Wk.	Dates	Topic(s)	Reading (Beachy—Blair)
1	1	Jan 15-19	Course introduction, quadratic equations, and complex numbers	Sec. 4.1 and Sec. A.5
	2	Jan 22-26	Polynomial algebra, roots, and factorization	Sec. 4.1-4.3.
	3	Jan 29 - Feb 02	Field isomorphisms and factor fields	Sec. 4.3 and Sec. 8.5
	4	Feb 05-09	Factor fields, integer polynomials and factorization tests	Sec. 4.3-4.4
2	5	Feb 12-16	Ring theory and polynomial rings	Sec. 5.1-5.2.
	6	Feb 19-23	Algebraic numbers and degrees	Sec. 6.1-6.2.
	7	Feb 26 - Mar 01	Algebraic field extensions and constructible numbers	Sec. 6.2-6.3.
	8	Mar 04-08	Transcendental numbers (flex topic!)	TBD
3	9	Mar 11-15	Splitting fields, finite fields, and automorphisms	Sec. 6.4
	10	Mar 18-22	Spring break	
	11	Mar 25-29	Uniqueness of splitting fields and structure of finite fields	Sec. 6.4-6.5
	12	Apr 01-05	Automorphisms of finite fields and numerical Galois theory	Sec. 8.1
4	13	Apr 08-12	Galois groups, fixed fields and primitive elements	Sec. 8.2-8.3.
	14	Apr 15-19	Galois correspondence	Sec. 8.3 and Sec. 8.5
	15	Apr 22-26	Solvable groups and radical extensions	Sec. 7.6 and Sec. 8.4
	16	Apr 29 - May 03	Insolubility of quintic	Sec. 8.4

Icon Credits

Revision Example

Revising *select all that apply* questions requires explaining both the logic behind the "right" answers and the logic behind the "wrong" answer. Let us see how that looks.

Question. (Select all that apply) Which of the following "theorems" from set theory are true?

(a) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(b) If $A \subseteq B$ and A and B are both countable, then $A = B$.

(c) If $A \subseteq B$ and A and B are both finite of the same size, then $A = B$.

Revised answer. The true statements are (a) and (c), but (b) is false. First, we explain why (a) is true. The statement $A \subseteq B$ says every element of A is in B. The statement $B \subseteq A$ says every element of B is in A. Therefore, if both $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, which means $A = B$.

Second, we explain why (c) is true. Suppose $A \subseteq B$ and $|A| = |B|$. We can write $B = A \cup C$ where $C = B \backslash A$ is the complement of A within B. The union is disjoint and so $|B| = |A| + |C|$. Since we are given that $|B| = |A|$, we see $|C| = 0$. Thus C is empty, and we conclude $A = B$ again.

Third, we explain why statement (b) is false with a counter-example. Let $A = \mathbb{Z}$ and $B = \mathbb{Q}$. Then $A \subseteq B$ and both are countable sets, but $A \neq B$ because $1/2 \in \mathbb{Q}$ but $1/2 \notin \mathbb{Z}$.

Tip. To show something is *false*, a counter-example usually suffices.

Grading Example

Here we provide a representative sample of how some rubric items will be applied in grading the Performance Tasks and/or Major Works.

Student writing.

Let G be a group and $a, b, g \in G$. Assume $a = gbg^{-1}$. Then

$$a = gbg^{-1} = gg^{-1}b = b.$$

Therefore, $a = b$.

Rubric application.

This fails "*Are relevant tools (definitions, techniques, axioms) accurately used?*" because in a general group, there is no axiom that lets you say $bg^{-1} = g^{-1}b$. (That would *require* that G is abelian.)

Student writing.

We will show $x = \sqrt{2}$ is irrational. To do this, we consider $x^2 = 2$. Since x^2 is an integer, its square root must be irrational.

Rubric application.

This fails "*Is a viable strategy to complete the Task clearly identified?*" because the strategy of squaring x to show x is irrational will not work without major modification. It will also fail "*Are inaccuracies limited to insubstantial details? (Including theory.)*" because the writing implicitly uses the False Theorem® "The square root of an integer is irrational."

Grading, cont.

Here are a few more examples of how the rubric for Performance Tasks can be applied.

Student writing.

We need to check $x^2 - 3x + 2$ factors over the rational numbers. Indeed,

$$x^2 - 3x + 2 = (x + 1)(x + 2)$$

and both factors $x + 1$ and $x + 2$ are rational polynomials.

Rubric application.

This passes "*Are inaccuracies limited to insubstantial details? (Including theory.)*" In this case, there is a small arithmetic inaccuracy because the linear factors should really be $x - 1$ and $x - 2$. But, that kind of error does not substantially impact the answer.

Student writing.

Consider the group $\mathbf{Z}_{12}$ of integers modulo 12 and $a = 7$ in $\mathbf{Z}_{12}$. Then a is a generator.

Rubric application.

This fails "*Is the task successfully completed with clear logic?*" because there is a logical gap, even if the writing is technically correct. An explanation for why 7 is a group generator of $\mathbf{Z}_{12}$ is missing!

Tip. You should add details if your writing is inaccurate without them. Above, if I replace $a = 7$ with $a = 4$, then the statement becomes false (since $4\mathbf{Z}_{12} = \{0,4,8\} \neq \mathbf{Z}_{12}$). So, something about $a = 7$ is missing and should be included!

Student writing.

The set of rational numbers is a subgroup of the real numbers. Indeed, if $a, b, c \in \mathbf{Q}$ then

$$(a + b) + c = a + (b + c).$$

The other subgroup axioms can also be checked.

Rubric application.

This fails "*Is the primary content identified and clearly addressed?*". The content of checking something is a subgroup is to check the subgroup axioms, which the student has explicitly skipped. Even worse, the "associative" axiom that is checked is the only group axiom that is automatic to check for a subgroup!

Student writing.

The rational numbers are a subgroup of the real numbers. Indeed, if $a = m/n \in \mathbf{Q}$ and $b = p/q \in \mathbf{Q}$ then

$$a + b = \frac{mp + np}{nq} \in \mathbf{Q}.$$

So $\mathbf{Q}$ is closed for $+$. We also have to check the identity and inverse axioms…

Rubric application.

Assuming the identity and inverse axioms are checked, this passes "*Is the primary content identified and clearly addressed?*" even if the associativity is not mentioned. That's because among the axioms for a subgroup, the primary things to check are the closure, identity, and inverses. The associativity is not primary because it is automatically inherited from the ambient group.