On the variation of (φ, Γ)-modules over p-adic families of automorphic forms

John Bergdall

Contents

Acknowledgements

I thank Joël Bellaïche for his unwavering support and guidance as my thesis advisor. I am grateful for his constant encouragement and the example he set for me while deftly attending to my education during my time at Brandeis. I hope that I am able to pay forward a small amount of the enormous debt which I owe him.

I thank Sheldon Joyner and Jay Pottharst for serving on my defense committee and for a number of helpful suggestions regarding this thesis.

Without the support, energy and openness of an insatiably curious mathematics community in Boston and around the world my research would have suffered. To the faculty, all the students and especially my close friends at Brandeis, my time spent there is fondly remembered. I am lucky to include in the latter group my entire class. Five years was too short for our time together.

For many helpful and inspiring conversations throughout the years, I thank Marco Aldi, Rebecca Bellovin, Thomas Barnet-Lamb, Shaunak Deo, Yu Fang, David Geraghty, Dmitry Kleinbock, Chan-Ho Kim, Brandon Levin, Paul Monsky, Susan Parker, Rob Pollack, Joe Rabinoff, Ila Varma, Jared Weinstein and Yurong Zhang. In this regard, a special thanks goes to Anna Medvedovsky for her exceptional patience and friendship. I hope that she has learned as much from me as I have from her. I must also acknowledge Kiran Kedlaya, Jay Pottharst and Liang Xiao for enthusiastically sharing their work with me in the seminars at Boston University and MIT, and in many conversations over last eighteen months.

Finally, it is impossible to imagine surviving to this point without the constant support of my family. Their encouragement and unconditional support provides me with the freedom to pursue any and every goal to which I aspire.

Introduction

We present, in this thesis, new results on the variation of Galois representations in p-adic families of automorphic representations. As an application of our main result, we obtain new smoothness results for such families. The theme we wish to emphasize is the confluence between analytic variation and infinitesimal deformations of Galois representations. Both themes have been exploited to resolve central conjectures in algebraic number theory such as: the Iwasawa main conjecture (for totally really fields and GL_2), Fermat's Last Theorem and the two-dimensional Fontaine-Mazur conjecture.

Background and history

In order to do justice the circle of ideas we are exploring, we humbly give a brief, and personal, recollection of the history and influences of the work contained herein.

p-adic families of modular forms from Serre to Coleman-Mazur. The genesis of modular forms in p-adic families parameterized by their weight goes back to the work of Serre [**58**] wherein the p-adic family of Eisenstein series was explained. Serre's families arose as formal q-expansions whose coefficients are p-adic limits of the coefficients of q-expansions associated to classical modular forms. A geometric construction of p-adic modular forms was given at the same time by Katz [**39**]. The arithmetic application, hinted at in Serre's original article, was that congruences between modular forms (required to build a p-adic family) can detect, and are explained by, congruences between special values of L-functions.

This tradition has been expanded upon greatly beginning with the early work of Hida [**37**]—the main difference between Hida's work and his predecessors is that he began focusing on congruences between (ordinary) eigenforms, rather than all modular forms. By the work of Eichler, Shimura and Deligne [**25, 59, 24**], one can associate a p-adic Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ to each eigenform f of weight $k \geq 2$. Here $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is the absolute Galois group of $\mathbf{Q}$. Thus, Hida's theory of p-adic families of ordinary eigenforms also gives rise to the construction [**38**] of what we now would understand to be a p-adic family of Galois representations. One important feature of ordinary eigenforms is that although the associated linear representations ρ_f are often irreducible, Wiles showed [**64**] that they become reducible upon restricting to a decomposition group at p. More specifically, if we denote by $\rho_{f,p}$ the restriction to $G_{\mathbf{Q}_p}$ then $\rho_{f,p}$ contains a one-dimensional subrepresentation on which the inertia group acts through a finite quotient. Such Galois representations are called ordinary. In fact, Mazur and Wiles [**50**] also explained that the "big" representations Hida constructed have this property over an entire p-adic family. That is, when you p-adically interpolate ordinary eigenforms you obtain a p-adic family of ordinary Galois representations.

In the seminal work [**48**], Mazur sought to develop an *a priori* framework in which one could discuss (infinitesmial) families of Galois representations. A reminder of Mazur's theory will occur in Chapter 3. One of the most striking results was the first example of an "$R = \mathbf{T}$" theorem. Loosely speaking, Mazur defined and studied a certain ring R which parameterized "ordinary deformations" of the mod p representation $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$. He then showed that all such deformations occur in a Hida family containing the given form f. A key ingredient is necessarily the result of Mazur-Wiles discussed above—this forms the $\mathbf{T}$-side of the picture. The other ingredient is precise calculations (of a group cohomological nature) inside the deformation ring R. It bears mentioning that further "$R = \mathbf{T}$" results were also at the heart of the proof of Fermat's Last Theorem [**65, 62**].

The extension of Hida's theory to non-ordinary cuspforms was an important goal in algebraic number theory during the final decade of the 20th century. The hint that such a theory existed was already present in Mazur's deformation theory and the work of Gouvêa and Mazur [**31**]. Ultimately, it was Coleman's development [**22**] of the spectral theory of compact operators acting on p-adic Banach spaces (Gouvêa's thesis [**32**] also contains early versions of these ideas) which proved to be the key technological input. In the fundamental work [**20**], Coleman and Mazur applied Coleman's theory to Katz's spaces of p-adic modular forms and constructed what we now know as an *eigencurve*. If we fix an integer N with $(N, p) = 1$ then the eigencurve of tame level N is a rigid analytic curve parameterizing what are known as finite slope overconvergent eigenforms—the p-adic interpolation of classical eigenforms on $\Gamma_1(N) \cap \Gamma_0(p)$. Strictly speaking, by eigenform here we mean an eigenvector for Hecke operators T_ℓ with $\ell \nmid Np$ and the Atkin-Lehner U_p-operator. We have as well that the eigencurve comes equipped with a family of Galois representations obtained by interpolating the representations ρ_f. It is natural to wonder if there is a result analogous to Mazur-Wiles for arbitrary points of the eigencurve. Notice that if f is an eigenform which is not ordinary at p then the representation $\rho_{f,p}$ is irreducible. Thus, any property of $\rho_{f,p}$ we hope to interpolate over an eigencurve must be hidden deeper within the structure of $\rho_{f,p}$.

Kisin's lemma and trianguline representations. We now recall Kisin's analog to Mazur-Wiles. We fix an integer N with $(N, p) = 1$ and denote by X the eigencurve of tame level N. Let $f = \sum a_n(f)q^n$ be an eigenform of level $\Gamma_1(N)$, weight $k \geq 2$ and nebentypus ε_f. Then, there are two points $x_{f,\alpha}$ and $x_{f,\beta}$ on X, one for each root of the polynomial

$$T^2 - a_p(f)T + p^{k-1}\varepsilon_f(p) = (T - \alpha)(T - \beta). \tag{0.1}$$

Moreover, we have that $T_\ell(x_{f,\alpha}) = T_\ell(x_{f,\beta}) = a_\ell(f)$ and $U_p(x_{f,r}) = r$ for $r = \alpha, \beta$. The interpolation of the Galois representations on X is specified at these points by setting $\rho_{x_{f,\alpha}} := \rho_f =: \rho_{x_{f,\beta}}$. Next, by work of Saito [**56**] one has that $D^+_{\mathrm{cris}}(\rho_{x_{f,r}})^{\varphi=r} \neq 0$ for either root $r = \alpha, \beta$. Here, D_{cris} is Fontaine's crystalline functor in p-adic Hodge theory and the $+$-superscript refers to the non-negative part of the Hodge filtration.

Kisin's major coup [**43**] was that the same is true at any point on the eigencurve: if ρ_x is the Galois representation at a point x on the eigencurve then

$$D^+_{\mathrm{cris}}(\rho_{x,p})^{\varphi=U_p(x)} \neq 0. \tag{0.2}$$

This vastly generalizes the result of Mazur-Wiles (that result is obtained by considering the unit root of an ordinary eigenform). As an application, Kisin was also able to deduce that the "geometric condition" of the Fontaine-Mazur conjecture [**30**] is enough to distinguish (most) classical eigenforms from non-classical ones on the eigencurve. He further showed that a certain universal deformation ring of Galois representations is the same as the (completed) local ring on the eigencurve. The missing cases[1] alluded to occur at classical points $x_{f,\beta}$ where f is ordinary, $v_p(\beta) = k-1$ and $f = \left(q\frac{d}{dq}\right)^{k-1} g$ for some overconvergent eigenform g of weight $2 - k$.

In order to study the higher-dimensional families of Galois representations we will make use of Colmez's fundamental notion of trianguline representations [**23**]. To explain this, we recall that

[1] The deformation-theoretic situation at these points have since been explained by Bellaïche [**3**] (see the precursor [**4**] as well).

attached to any p-adic Galois representation ρ_p of $G_{\mathbf{Q}_p}$ one can associate a (φ,Γ)-module $D_{\mathrm{rig}}(\rho_p)$ over the Robba ring (these terms will be explained in Chapter 2). For now, it suffices to say that a (φ,Γ)-module is a finite free module over a certain commutative ring with semi-linear actions of an operator φ and a group Γ. Some, but not all, (φ,Γ)-modules are of the form $D_{\mathrm{rig}}(\rho_p)$. Nevertheless, many of the tools we use to study Galois representations (p-adic Hodge theory, Galois cohomology, etc.) extend to the category of (φ,Γ)-modules.

We say that a (φ,Γ)-module is *trianguline* if it is upper triangular with respect to the operator φ and the group of operators Γ. A representation ρ_p is a *trianguline representation* if $D_{\mathrm{rig}}(\rho_p)$ trianguline; the choice of a filtration making it upper triangular is called a triangulation. Note, any crystalline representation is trianguline (in many different ways). In these terms, Kisin's result (0.2) is rephrased by saying that the representations on the eigencurve are all trianguline. That is, for each x on the eigencurve X there is a rank one (φ,Γ)-module $L_x \subset D_{\mathrm{rig}}(\rho_{x,p})$. Moreover then, his deformation-theoretic result says that L_x varies analytically with x away from a discrete set. Specifically, we mean away from points $x_{f,\beta}$ where f is ordinary of weight k, $v_p(\beta) = k-1$ and $f = (q\frac{d}{dq})^{k-1}g$ for an overconvergent eigenform g of weight $2-k$. Such points are known as *critical points* and their higher-dimensional analogues are the main objects of this thesis.

Statement of results and methods

The higher-dimensional analogs of the eigencurve are known as *eigenvarieties*—they are obtained as spaces of p-adic automorphic forms for a connected reductive group G. The applications in Chapter 5 will occur in the case of definite unitary groups attached to an imaginary quadratic extension $E/\mathbf{Q}$ such that such that $G(\mathbf{Q}_p) \cong \mathrm{GL}_n(\mathbf{Q}_p)$. In particular, we assume that p splits in E. Let us fix such a group (and quadratic extension) for concreteness.

Historically, there are multiple creators of eigenvarieties in this context. For example, Chenevier [**15**] and Loeffler [**47**] have each provided a construction using Buzzard's eigenvariety machine [**12**]. On the other hand, a different, more general, construction is given by Emerton [**26**] using completed cohomology. The common theme is that one considers a fixed class of regular, algebraic automorphic representations for G and their systems of eigenvalues for an associated Hecke algebra. An eigenvariety is then obtained by p-adically interpolating the eigenvalues at these "classical points". The resulting rigid analytic space (which should be equidimensional of dimension n) is independent of its creator and we refer to any of them as an eigenvariety. We will intentionally stay vague for the rest of the introduction as to what class of representations one considers—it suffices for now to say that we will consider representations π such that π_p is an unramified representation of $\mathrm{GL}_n(\mathbf{Q}_p)$.

Eigenvarieties also come equipped with a natural family of Galois representations. Indeed, by the work of many people we now know how to attach a Galois representation $\rho_\pi : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ to any algebraic automorphic representation π for G. Here, S is a finite set of places of E depending only on G and the class of representations we are interpolating. The interpolation of these representations over the eigenvariety provide us with a rich example of a p-adic family of Galois representations. The questions which this thesis seeks to answer are:

(Q1) What is the analog of the results of Mazur-Wiles and Kisin?
(Q2) Are the local rings on eigenvarieties also naturally deformation rings for Galois representations?

Previous work. The answers to the questions (Q1) and (Q2) have been given by Bellaïche-Chenevier [**5**] in a generic situation on eigenvarieties. The representations ρ_π are crystalline at p (because π_p is unramified) and thus have a triangulation. Moreover, the classical points on eigenvarieties actually correspond to pairs (π, P) where π is an automorphic representation for G

and P is a triangulation of $\rho_{\pi,p}$—this is the analog of choosing a root of (0.1) in the case of modular forms. Thus, though we expect $\rho_{\pi,p}$ has many triangulations, there is a canonical triangulation at each corresponding point on an eigenvariety.

Fix a point (π, P) and consider the representation $\rho_{\pi,p}$. By definition, P defines a filtration

$$0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n = D_{\mathrm{rig}}(\rho_{\pi,p}) \tag{0.3}$$

where each P_i is a (φ, Γ)-submodule of rank i and the quotients are all (φ, Γ)-modules as well. We attach to this triangulation an ordered list of integers $(s_1, \ldots, s_n)$ by specifying that the Hodge-Tate weights of P_i be $\{s_1, \ldots, s_i\}$. Then, we say that P is a non-critical triangulation of $\rho_{\pi,p}$ if the sequence $(s_1, \ldots, s_n)$ of weights is strictly increasing. This is a Zariski open and dense condition within the space of all classical points (π, P) on an eigenvariety. The answers given by Bellaïche-Chenevier over the non-critical locus are:

(Q1) The triangulations (0.3) extend to formal neighborhoods around non-critical points.
(Q2) Under some techincal hypothesis, the local rings on "minimal eigenvarieties" parameterize deformations of the global representations ρ_π which are:
- conjugate self-dual (an Archimedean condition),
- unramified away from p (in the sense of Bloch-Kato), and
- trianguline at p.

Here, an infinitesimal deformation of ρ_p is called trianguline if it has a triangulation deforming the given one (0.3). We will discuss what we mean by "minimal" below, following Theorem B. We will give, as well, some remarks on their work when we compare and contrast our methods below.

The results of this thesis. For simple reasons (e.g. coming from Sen's theory in p-adic families) one knows that neither answer given by Bellaïche-Chenevier can be correct at critical points. Instead, we have to relax our expectations about which pieces of the triangulation we expect to analytically continue in formal neighborhoods of points. Assume that $\rho_{\pi,p}$ has Hodge-Tate weights $k_1 < \cdots < k_n$. Let us call a step P_i in (0.3) *non-critical* if we have $\{s_1, \ldots, s_i\} = \{k_1, \ldots, k_i\}$. We then can consider the filtration

$$0 \subsetneq P_{i_1} \subsetneq \cdots \subsetneq P_{i_{s-1}} \subsetneq P_{i_s} = D_{\mathrm{rig}}(\rho_{\pi,p}) \tag{0.4}$$

where P_{i_j} appears if and only if P_{i_j} is non-critical. Following Chenevier's terminology in [**16**], we call such a filtration a *parabolization*. The main techincal result of this thesis, and our answer to (Q1), is the following theorem. The theorem can be found in the text as Theorem 4.13—there are more precise hypotheses listed there as well.

Theorem A. *Suppose that (π, P) is a classical point on an eigenvariety and ρ_π is irreducible. Then, the parabolization* (0.4) *analytically continues in an affinoid neighborhood of the point (π, P).*

Notice that our result is valid over entire affinoid neighborhoods rather than just finite thickenings. Thus, even at non-critical points (where (0.4) is the same as (0.3)) we have strengthened the results of Bellaïche-Chenevier and Kisin, at least in the case that the global representation ρ_π be irreducible.

Our answer to the question (Q2) requires slightly more explanation than (Q1). We denote by $\widetilde{\rho}$ an infinitesmial deformation of $\rho_{\pi,p}$. We say that $\widetilde{\rho}$ is a *paraboline deformation* with respect to a parabolization (0.4) if there is a parabolization of $D_{\mathrm{rig}}(\widetilde{\rho})$ deforming the given one. One should note that this is not necessarily even a condition on $\widetilde{\rho}$. Indeed, it can happen that the ordering $(s_1, \ldots, s_n)$ of the weights associated to the triangulation (0.3) is given by by $s_1 = k_n, s_2 = k_{n-1}, \ldots, s_n = k_1$. In that case the parabolization (0.4) is just the trivial parabolization and thus, to ask for a paraboline deformation is the same as to ask for a deformation.

We remedy the possible lack of content by passing further into the theory of (φ,Γ)-modules. Suppose that D is any crystalline (φ,Γ)-module and we fix a triangulation

$$0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_t = D \tag{0.5}$$

of D. Since being crystalline is stable under subquotient, each P_i is crystalline. We denote by $(s_1,\ldots,s_t)$ the ordering of the Hodge-Tate weights of D determined by the triangulation (0.5) and by ϕ_i the crystalline eigenvalue of the rank one crystalline (φ,Γ)-module P_i/P_{i-1}. For reasons made clear in the text, we make a re-normalization and denote $F_i = p^{-s_i}\phi_i$. It is easy to see that for each $i=1,\ldots,t$ one has that

$$D^+_{\text{cris}}(\wedge^i D(s_1+\ldots+s_i))^{\varphi=F_1\cdots F_i} \neq (0).$$

Now we consider a deformation $\widetilde{D}$ of D. For each $i=1,\ldots,t$ we denote by $\widetilde{s}_i$ the Hodge-Tate-Sen weight of $\widetilde{D}$ which deforms s_i. We then say that $\widetilde{D}$ is a *Kisin-type* deformation with respect to (0.5) provided for each $i=1,\ldots,t$ the module $D^+_{\text{cris}}(\wedge^i\widetilde{D}(\widetilde{s}_1+\cdots+\widetilde{s}_i))^{\varphi=\widetilde{F}_1\cdots\widetilde{F}_i}$ is free of rank one for some choice of elements $\widetilde{F}_i$ deforming the F_i.

Return now to the linear representation $\rho_{\pi,p}$. We consider all deformations $\widetilde{\rho}$ of $\rho_{\pi,p}$ such that the following hold.

- The deformation $\widetilde{\rho}$ is paraboline with respect to the parabolization (0.4). We denote by
$$0 \subsetneq \widetilde{P}_1 \subsetneq \widetilde{P}_2 \subsetneq \ldots \subsetneq \widetilde{P}_s = D_{\text{rig}}(\widetilde{\rho})$$
the deformation of the parabolization.
- Each of the (φ,Γ)-modules $\widetilde{P}_j/\widetilde{P}_{j-1}$ is a Kisin-type deformation of $P_{i_j}/P_{i_{j-1}}$ with respect to the induced triangulation coming from (0.3).

Such a deformation $\widetilde{\rho}$ we coin as a *Kisin-type paraboline deformation*. It is not too hard to see that paraboline deformations of Kisin-type form a relatively representable subfunctor of all deformations of $\rho_{\pi,p}$. The Kisin-type paraboline deformation conditions leads to our answer of (Q2).

Recall that we have fixed π and a corresponding point (π,P) on an eigenvariety. We have as well the linear representation ρ_π. Assume that ρ_π is irreducible—this guarantees a satisfactory deformation theory. We denote by $R_{\rho_\pi,P}$ the universal deformation ring of ρ_π parameterizing deformations $\widetilde{\rho}$ which are

- conjugate self-dual (again, an Archimedean condition)
- unramified away from p (in the sense of Bloch-Kato), and
- $\widetilde{\rho}_p := \widetilde{\rho}\big|_{G_{\mathbf{Q}_p}}$ is a Kisin-type paraobline deformation of $\rho_{\pi,p}$.

The ring $R_{\rho_\pi,P}$ is a complete local noetherian ring and we denote by $\mathfrak{t}_{R_{\rho_\pi,P}}$ its Zariski tangent space. We denote as well the completion of the rigid local ring at the point (π,P) by $\widehat{\mathcal{O}}^{\text{rig}}_{(\pi,P)}$ and $\mathfrak{t}_{(\pi,P)}$ the Zariski tangent space at (π,P).

Theorem B. *Assume that the eigenvariety is "minimal" for π. Then, one has a natural surjection*

$$R_{\rho_\pi,P} \twoheadrightarrow \widehat{\mathcal{O}}^{\text{rig}}_{(\pi,P)} \tag{0.6}$$

and an inequality $n \leq \mathfrak{t}_{(\pi,P)} \leq \mathfrak{t}_{R_{\rho_\pi,P}}$.

The term minimal here refers to issues away from p but inside the set of places where ρ_π could ramify. In the case of the eigencurve, if f is a newform at level N then the tame level N eigencurve would be minimal for f. The inequalities at the end of Theorem B are formal from (0.6). The case where we can show that each inequality is an equality is the following and gives a complete answer to (Q2).

THEOREM C. *Assume that ρ_π is irreducible and that (π, P) denotes a point on a minimal eigenvariety for π. Assume, as well, that*

- $H^1_f(G_{E,S}, \operatorname{ad}\rho_\pi) = (0)$, *and*
- *the Kisin-type deformation problems on the associated gradeds of the parabolization* (0.4) *satisfy a hypothesis* (3.15).

Then, the map (0.6) *is an isomorphism and $\widehat{\mathcal{O}}^{\mathrm{rig}}_{(\pi,P)}$ is a regular local ring of dimension n.*

Let us remark quickly on the two hypotheses. First, H^1_f is the "fine" Bloch-Kato Selmer group defined in [**10**]. This particular vanishing is a techincal hypothesis which can be shown in many cases. Moreover, it is conjectured to always to be true. The second hypothesis, on the other hand, should be treated (at this point of our knowledge) as being due to limitations in our ability to compute Kisin-type deformation rings with respect to fully critical triangulations. For example, it is shown in §3.3 that the second hypothesis is satisfied if each graded in (0.4) has rank at most two. In particular, this gives completely new results at many critical points. It includes, as well, the answer to (Q2) at non-critical points given in [**5**].

Methods. The main place where we have developed new methods is in the proof of Theorem A, but let us first explain how to deduce the final two results given Theorem A.

In order to deduce Theorem B from Theorem A we follow closely the ideas of Mazur and Kisin (in the case of modular forms) and Bellaïche-Chenevier in general. The hypothesis that ρ_π be irreducible first guarantees the existence of the deformation ring $R_{\rho_\pi,P}$. Furthermore, it also implies that one has a deformation $\widehat{\rho}_\pi : G_{E,S} \to \mathrm{GL}_n(\widehat{\mathcal{O}}^{\mathrm{rig}}_{(\pi,P)})$ of the representation ρ_π (this is a result of Rouquier and Nyssen). Thus, to show that we have a map (0.6) we need to show that $\widehat{\rho}_\pi$ has the three properties which define $R_{\rho_\pi,P}$. It is not hard to see that $\widehat{\rho}_\pi$ must be conjugate self-dual because it is true at every classical point. The deformation conditions away from p are valid for $\widehat{\rho}_\pi$ as we work on a minimal eigenvariety for π and because of recent advances in local-global compatibility for the representation ρ_π. Finally, $\widehat{\rho}_{\pi,p}$ is a Kisin-type paraboline deformation because of Theorem A. Thus we earn a map (0.6) and the construction of the eigenvariety implies easily that the map is surjective. We have already remarked that the inequalities are formal.

Now, once we have Theorem B in hand we see that in order to deduce the result of Theorem C we must show that $\dim \mathfrak{t}_{R_{\rho_\pi,P}} \leq n$. The hypothesis on H^1_f and the deformation conditions away from p imply that there is an inclusion

$$\mathfrak{t}_{R_{\rho_\pi,P}} \hookrightarrow \mathfrak{t}_{R_{\rho_{\pi,p},P}}/H^1_f(G_{\mathbf{Q}_p}, \operatorname{ad}\rho_{\pi,p}). \tag{0.7}$$

Here, $\mathfrak{t}_{R_{\rho_{\pi,p},P}}$ is the Zariski tangent space for the deformations of $\rho_{\pi,p}$ which are Kisin-type paraboline deformations. This insight is really due to Bellaïche-Chenevier. The goal now is to make a Galois cohomology calculation and show that the quotient on the right hand side is at most n-dimensional. The paraboline part of the calculation is easy and conceptual. The Kisin-type part, while not hard, is "by hand". The second hypothesis in Theorem C reflects all the cases where we have carried out this computation. Notice that in the non-critical case, there are no "by hand" calculations.

It remains to explain how we deduce Theorem A. Here the story diverges quite a bit from previous work. The first step is to construct a (φ, Γ)-module D over an affinoid neighobrhood near a point (π, P). For this, we first use that ρ_π is irreducible to extend the representation to an affinoid neighborhood and then we take the associated (φ, Γ)-module (constructed in this generality by Kedlaya and Liu).

In order to construct parabolizations of D we must first understand possible candidates for the terms of the parabolization over the family—among the terms of (0.5), only the entire module *a priori* extends to an affinoid neighborhood. The second issue is the actual construction of the parabolizations. Both issues are almost taken care of by the construction of the eigenvariety. Indeed, the construction tells us exactly what the associated gradeds of an affinoid-local triangulation should be in the case of a non-critical point. Thus, we could try to construct a triangulation over the entire base by repeatedly attempting to construct rank one submodules of quotients of D. In carrying out the process, we make use of recent results of Kedlaya, Pottharst and Xiao [**41**] on the Galois cohomology of (φ, Γ)-modules over affinoid bases. Using their results, we erect a standard cohomology and base change archetype from which we can deduce the existence of maps between (φ, Γ)-modules over affinoid bases from pointwise information on a Zariski dense set. The pointwise information is given, in turn, by studying the Galois cohomology of (φ, Γ)-modules over a field [**23, 46**].

This method works perfectly at a non-critical point. There, we inductively construct rank one submodules of quotients of D. At each step, the quotient we construct remains projective and we happily continue on with the induction. However, it is seen to break down at a critical point. Indeed, even after the first step, one notices that the quotients being constructed need not be projective anymore and thus we have left the world of (φ, Γ)-modules. Nevertheless, we carry on the process working instead with *generalized* (φ, Γ)*-modules* over affinoid bases. Since the main result of Kedlaya-Pottharst-Xiao extends to this larger class of objects and we can still use it to construct filtrations inside these new modules. The rest of the game is then to determine when the generalized (φ, Γ)-modules one obtains are bona fide (φ, Γ)-modules. We show that it happens exactly at the non-critical steps in the process. Just as in the non-critical case, Galois cohomology plays a key role here.

Organization

This thesis is arranged in five chapters. The proofs of the three theorems mentioned can be found in Chapters 4 and 5.

The first chapter describes input needed from rigid geometry. This includes the construction of the Robba ring over an affinoid base and the definition of (generalized) (φ, Γ)-modules. We have been greatly inspired in this regard by [**41**, Chapter 2].

The second chapter covers the arithmetic theory of (φ, Γ)-modules. Our first task is to remind the reader of the dictionary between Galois representations and (φ, Γ)-modules. We include here the extension of p-adic Hodge theory from Galois representations to (φ, Γ)-modules. A significant amount of time is devoted to the main theorems on the Galois cohomology of (φ, Γ)-modules over a field given by [**23, 46**]. Finally, we develop the formalism of parabolizations of (φ, Γ)-modules following Chenevier [**16**, §3].

The third chapter represents our attempt to expose the deformation theory of (φ, Γ)-modules. The results here are formally similar to those of linear representations and so we move quickly in some places. We do pause and compute the Zariski tangent spaces of many deformation functors. These include the Bloch-Kato Selmer groups and paraboline deformation functors to be sure. We also give precise meaning to the deformation problems at p discussed in this introduction. The chapter ends with some calculations of Kisin-type deformation rings in the fully critical case, which we recall is the bottleneck in the applicability of Theorem C.

The fourth chapter is the technical heart of this work. It includes the extension mentioned in the previous section of [**41**] to generalized (φ, Γ)-modules as well as the proof of Theorem A. Here we work within the the framework of "refined families of (φ, Γ)-modules". There is some time spent, though not a lot, explaining their basic properties and what one can deduce from the definitions. Moving on, we prove Theorem A in three different levels of generality—the non-critical case, the "minimally critical case" and the general case. We hope that the reader daunted by the explosion of indices in the final proof will find peace in the first two.

This brings us to the final chapter. We first orient ourselves with an exposition of eigenvarieties. We do not include the construction but we do provide a leisurely explanation of the interaction between automorphic data and Galois representations. Here, as well, we go to great lengths to make sure we have the most up-to-date references on the construction of Galois representations attached to automorphic representations for definite unitary groups. With the definition and properties of eigenvarieties finished, we move on to stating and proving the two theorems Theorem B and Theorem C. Most of this time is spent explaining how the results of Chapter 4 can be applied to the eigenvariety formalism we develop, i.e. why eigenvarieties define refined families.

Notations and conventions

Throughout, we consider primes ℓ and p (possibly the same). We choose now and forever isomorphisms $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ and $\iota_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$.

The adeles. We use v to denote (possibly infinite) places of a number field. The adeles (over $\mathbf{Q}$) are the restricted direct product

$$\mathbf{A} = \prod_v{}' \mathbf{Q}_v$$

with respect to the open subgroups $\mathbf{Z}_\ell \subset \mathbf{Q}_\ell$ at finite primes ℓ. We denote by

$$\mathbf{A}_f = \prod_{v \nmid \infty}{}' \mathbf{Q}_v.$$

the finite adeles

ℓ-adic fields. By an ℓ-adic field we will always mean a finite extension of the field $\mathbf{Q}_\ell$ of ℓ-adic numbers. If E is a ℓ-adic field then we will always normalize the ℓ-adic valuation so that $|-|_\ell : E \to \mathbf{Q}$ has $|\ell|_\ell = 1$.

Galois groups. If E is a field and E^{sep} is the choice of a separable closure for E then we let $G_E := \mathrm{Gal}(E^{\mathrm{sep}}/E)$. This is a profinite group, an open system of neighborhoods of the identity being given by $\{G_{E'}\}$ where E'/E is a finite extension.

In the case that $E/\mathbf{Q}_\ell$ is an ℓ-adic field, we have a short exact sequence

$$1 \to I_E \to G_E \to G_k \to 1$$

where k is the residue field of E and I_E is the inertia subgroup of G_E. Since k is a finite field, we have that $G_k \cong \widehat{\mathbf{Z}}$, topologically generated by the automorphism $\sigma(x) = x^{\#k}$. The Weil group of E is the subgroup W_E of G_E mapping to the integer powers of σ, i.e. W_E fits into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_E & \longrightarrow & W_E & \xrightarrow{\|-\|} & \sigma^{\mathbf{Z}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_E & \longrightarrow & G_E & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

The top short exact sequence defines a topology on W_E (different than the induced topology from G_E). We use $\mathrm{Frob} \in W_E$ to denote lifts of the *geometric* Frobenius σ^{-1}.

Galois representations. Suppose that E is a field and L a topological field. Then, a Galois representation of G_E over L is a continuous homomorphism $\rho : G_E \to \mathrm{GL}_n(L)$. The special cases we will use are E is either a number field or an ℓ-adic field and L a p-adic field (possibly with $\ell = p$).

Suppose that E is any one of these fields. Choose a compatible system $\{\zeta_{p^n}\}_n$ of primitive roots of unity in $\overline{E}$. Then G_E acts on each root of unity by $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{n,\mathrm{cycl}}(g)}$ for some $\chi_{n,\mathrm{cycl}}(g) \in (\mathbf{Z}/p^n\mathbf{Z})^\times$. Taking $n \to \infty$ we obtain the p-adic cyclotomic character

$$\chi_{\mathrm{cycl}} : G_E \to \mathbf{Z}_p^\times.$$

If ρ is a p-adic representation then we use the notation $\rho(n)$ to denote the representation $\rho \otimes_{\mathbf{Q}_p} \chi_{\mathrm{cycl}}^n$.

p-adic Hodge theory. In the case that E and L are both p-adic fields use the standard functors of p-adic Hodge theory D_{Sen}, D_{dR}, etc. giving rise to representations which are Hodge-Tate, de Rham, etc. We adopt the convention that χ_{cycl} has Hodge-Tate weight -1.

Weil-Deligne representations. Let E be an ℓ-adic field. An n-dimensional Weil-Deligne representation is a pair (r, N) where

- $r : W_E \to \mathrm{GL}_n(\mathbf{C})$ is a continuous n-dimensional representation,
- $N \in M_N(\mathbf{C})$ is a nilpotent matrix, and
- for all $g \in W_E$, $r(g)^{-1} N r(g) = \# k^{\|g\|} N$.

Suppose now that L is a p-adic field and E is an ℓ-adic field for $\ell \neq p$. If $\rho : G_E \to \mathrm{GL}_n(L)$ is a continuous representation we use $\mathrm{WD}(\rho)$ to denote its associated Weil-Deligne representation.

Keep L as above but suppose that E is a p-adic field now. If ρ is de Rham then one can still attach a Weil-Deligne representation $\mathrm{WD}(\rho)$. In the case that E is crystalline $\mathrm{WD}(\rho) = D_{\mathrm{cris}}(\rho)$ with $\mathrm{Frob} \in W_L$ acting through the crystalline Frobenius φ.

Local class field theory. Suppose that E is an ℓ-adic field. Then, the local Artin map

$$\mathrm{Art}_E : E^\times \xrightarrow{\cong} W_E^{\mathrm{ab}}$$

is normalized so that a uniformizing parameter goes to a geoemtric Frobenius elements. In the case that $E = \mathbf{Q}_\ell$ we use the notation Art_ℓ.

The local Langlands correspondence. Suppose that E is an ℓ-adic field. The local Langlands correspondence defines a bijection

$$\left\{ \begin{matrix} \text{smooth representations} \\ \text{of } \mathrm{GL}_n(E) \end{matrix} \right\} \xrightarrow{\mathrm{rec}_F} \left\{ \begin{matrix} \text{Weil-Deligne representations} \\ (r, N) \text{ of } W_E \end{matrix} \right\}.$$

It is normalized so that if χ is a smooth character $E^\times \to \mathbf{C}^\times$ and π is any smooth representation of $\mathrm{GL}_n(E)$ then

$$\mathrm{rec}_E(\pi \otimes \chi(\det)) = \mathrm{rec}_E(\pi) \otimes (\chi \circ \mathrm{Art}_E^{-1})$$

CHAPTER 1

Affinoid algebras, the Robba ring and (φ, Γ)-modules

In this brief first chapter we recall the theory and conventions we will need from Tate's rigid geometry. Second, we remind the reader of the Robba ring over the field $\mathbf{Q}_p$. Third, we explain what we will mean by the Robba ring over a $\mathbf{Q}_p$-affinoid algebra—the key for studying (φ, Γ)-modules in families (see [**40, 41**]). We will spend some time recalling and developing the main objects we are going to study: finitely presented modules over the Robba ring with coefficients in an affinoid algebra. In the final section we elaborate on the construction of (φ, Γ)-modules in this setting. We end by revealing the connection between (φ, Γ)-modules and Galois representations.

1.1. Affinoid algebras (over $\mathbf{Q}_p$)

The standard theory of affinoid algebras is explained in [**11**]. We will (unless we change our mind, in which case we hopefully will remind the reader) use A to denote an affinoid $\mathbf{Q}_p$-algebra. That is, A is a quotient of a standard Tate algebra

$$\mathbf{Q}_p\langle T_1, \dots, T_r\rangle = \left\{ \sum_I a_I T^I \in \mathbf{Q}_p[[T_1, \dots, T_r]] \colon \, |a_I| \to 0 \text{ as } I \to \infty \right\}.$$

The usual Gauss norm passes to the quotient (any ideal of a Tate algebra is closed with respect to the Gauss norm) and gives A a norm as well. Associated to any such A, we have the rigid space $X := \mathrm{Sp}(A)$ whose points are in bijection with the closed points of $\operatorname{Spec} A$ (i.e. the maximal ideals of A). We denote by $\mathbf{B}^r$ the rigid space $\mathrm{Sp}(\mathbf{Q}_p\langle T_1, \dots, T_r\rangle)$ which is our model of the closed disc of radius one. We should not have to deal deeply with the affinoid topology, which of course is *not the same as the Zariski topology.* If x is a point of X corresponding to the maximal ideal $\mathfrak{m}_x$, we use the notation $\mathcal{O}_{X,x}^{\mathrm{rig}}$ to denote the local ring of the space X at the point x. We will also write $\mathcal{O}_{X,x}$ to denote the algebraic localization $A_{\mathfrak{m}_x}$. Each are local rings, the ring $\mathcal{O}_{X,x}^{\mathrm{rig}}$ is Henselian and their completions at their maximal ideals are naturally isomorphic.

If $x \in X$ corresponds to a maximal ideal $\mathfrak{m}_x \subset A$ then we will denote *the residue field at* x by $L(x) := A/\mathfrak{m}_x$. By our previous comments this is the same as the residue field of $\mathcal{O}_{X,x}^{\mathrm{rig}}$, which is also the same as the residue field of $\widehat{A}_{\mathfrak{m}_x}$. If $f \in A$ then we can take its value in the residue field $f(x) := f \bmod \mathfrak{m}_x$ without reference to which topology we consider.

Most of our work is going to concern modules over affinoid algebras but with reference to their properties in the Zariski topology. In particular, we will be mostly interested in the ranks of finite A-modules. If M is a module over A and x corresponds to a point of X then we denote

$$M_x := M \otimes_A L(x) = \left(M \otimes_A \mathcal{O}_{X,x}^{\mathrm{rig}} \right) \otimes_{\mathcal{O}_{X,x}^{\mathrm{rig}}} L(x) = (M \otimes_A \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} L(x).$$

The notation extends to sheaves over a general rigid space. The following result allows us to deduce statements about A-modules by only considering the points of X.

PROPOSITION 1.1 ([**11**, Proposition 6.1.1/3]). *If A is an affinoid algebra then it is a Jacobson ring. In particular, the subset* $\mathrm{Sp}(A) \subset \mathrm{Spec}(A)$ *is Zariski dense.*

We now give results in the theory of coherent sheaves over rigid spaces (as explained in [**11**, §9.4]) that need proving since we cannot find a reference for them. Recall that if A is an affinoid algebra then to any A-module M there is a canonically associated sheaf $\widetilde{M}$. Moreover, Kiehl's theorem [**11**, Theorem 9.4.3/3] says that the functor $M \mapsto \widetilde{M}$ defines an equivalence between finite A-modules and coherent sheaves on $\mathrm{Sp}(A)$.

LEMMA 1.2. *Suppose that $X = \mathrm{Sp}\, A$ and $Y = \mathrm{Sp}\, B$ are affinoid algebras over $\mathbf{Q}_p$. Then, the projection $X \times_{\mathbf{Q}_p} Y \to X$ is surjective and closed (in either the Zariski or the rigid topology).*

PROOF. Recall that $X \times_{\mathbf{Q}_p} Y = \mathrm{Sp}(A \widehat{\otimes}_{\mathbf{Q}_p} B)$. The fact that the map is surjective is clear, being the base change of a surjective map $Y \to \mathrm{Sp}(\mathbf{Q}_p)$. To see that it is closed, choose r and s so that we have a diagram

$$\begin{array}{ccc} X \times_{\mathbf{Q}_p} Y & \longrightarrow & \mathbf{B}^{r+s} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{B}^{r} \end{array}$$

whose horizontal arrows are closed immersions (see [**11**, 7.1.3] for a discussion). The closedness of the left vertical arrow then follows from the closedness of the right vertical arrow. □

Recall that if X is a topological space and $f : X \to \mathbf{N}$ is a function then f is upper semi-continuous if for all n, the set

$$\{x \in X : f(x) \geq n\}$$

is closed. In particular, if f is upper semi-continuous then its minimum is achieved on an open subset of X.

PROPOSITION 1.3. *Let $X = \mathrm{Sp}(A)$ as above.*

(a) If $\mathcal{Q}$ is a coherent sheaf on X then $x \mapsto \dim_{L(x)} \mathcal{Q}_x$ is upper semi-continuous in the Zariski (and thus affinoid) topology.

(b) Now suppose that Y is another rigid space over $\mathbf{Q}_p$. Let $\mathcal{Q}$ be a coherent sheaf on $X \times_{\mathbf{Q}_p} Y$ and assume that for all $x \in X$, $\mathcal{Q}_x$ is free over $Y_x := L(x) \times_{\mathbf{Q}_p} Y$. Then, the function $x \mapsto \mathrm{rank}_{Y_x} \mathcal{Q}_x$ is upper semi-continuous on X in the Zariski (and thus affinoid) topology.

PROOF. By Kiehl's theorem, $\mathcal{Q} = \widetilde{Q}$ for some finite A-module Q and $\mathrm{rank}_{L(x)} \mathcal{Q}_x = \mathrm{rank}_{L(x)} Q_x$. Since A is noetherian, Nakayama's lemma implies that $x \mapsto \dim_{L(x)} Q_x$ is upper semi-continuous on $\mathrm{Spec}(A)$. This proves (a). In the case of (b) we can assume without loss of generality that $Y = \mathrm{Sp}(B)$ is affinoid. Indeed, if $Y' \subset Y$ is open then since $\mathcal{Q}_x$ is free for each x we have $\mathrm{rank}_{Y'_x} \mathcal{Q}_x = \mathrm{rank}_{Y_x} \mathcal{Q}_x$. In this case $\mathcal{Q}$ is again associated to a finite $A \widehat{\otimes}_{\mathbf{Q}_p} B$-module Q. Then, to prove (b) it is enough to show that if $z \in X \times_{\mathbf{Q}_p} Y$ is a point lying over x then

$$\dim_{L(z)} Q_z = \mathrm{rank}_{L(x) \otimes_{\mathbf{Q}_p} B} Q_x. \tag{1.1}$$

Indeed, if π_X is the projection $X \times_{\mathbf{Q}_p} Y \to X$ then under (1.1) (and using the surjective part of Lemma 1.2) we have

$$\left\{ x \in X : \mathrm{rank}_{L(x) \otimes_{\mathbf{Q}_p} B} Q_x \geq n \right\} = \pi_X \left(\left\{ z \in X \times_{\mathbf{Q}_p} Y : \dim_{L(z)} Q_z \geq n \right\} \right).$$

Since π_X is closed (by Lemma 1.2), the right hand side is closed by part (a). Thus, the left hand side is also closed.

So, it just remains to explain (1.1). Let x and z be given with $\pi_X(z) = x$ and consider the maps

$$\{z\}^{\mathrm{red}} \to \{x\}^{\mathrm{red}} \times_{\mathbf{Q}_p} Y \xrightarrow{i_x \times \mathrm{id}} X \times_{\mathbf{Q}_p} Y$$

whose composition we denote by j_z. By assumption we have that $(i_x \times \mathrm{id})^* Q$ is free of rank equal to $\mathrm{rank}_{L(x)\otimes_{\mathbf{Q}_p} B} Q_x$. Thus, $j_z^* Q$ is a vector space over $L(z)$ with this dimenison. On the other hand, $j_z^* Q$ has dimension $\dim_{L(z)} Q_z$ as well. □

COROLLARY 1.4. *Suppose now that $X = \mathrm{Sp}(A)$ is reduced and that Y is another rigid space. Assume that $\mathcal{Q}$ is a coherent sheaf over $X \times_{\mathbf{Q}_p} Y$ such that for all $x \in X$ the fiber $\mathcal{Q}_x$ is free over $Y_x := L(x) \times_{\mathbf{Q}_p} Y$. Then, there exists an open affinoid $U \subset X$ such that $\mathcal{Q}\big|_{U \times_{\mathbf{Q}_p} Y}$ is finite projective. If $\mathrm{rank}_{Y_x} \mathcal{Q}_x$ is independent of $x \in X$ we can take $U = X$.*

PROOF. By Proposition 1.3(b) we can shrink X and assume that $x \mapsto \mathrm{rank}_{Y_x} \mathcal{Q}_x$ is constant on $X \times_{\mathbf{Q}_p} Y$. Furthermore, we can assume that Y is an affinoid $Y = \mathrm{Sp}(B)$. The result then follows from [**41**, Lemma 2.1.10(2)]—this is where we use that A is reduced. If $x \mapsto \mathrm{rank}_{Y_x} \mathcal{Q}_x$ is constant then we never had to change X. □

1.2. The Robba ring

We are now going to give the main examples of rigid spaces we will consider. They will be products of an affinoid base $X = \mathrm{Sp}(A)$ together with certain subdomains of the open p-adic unit disc. In the case where the affinoid base is a finite extension of $\mathbf{Q}_p$, we recall that such spaces behave essentially as though they are principal ideal domains (see Proposition 1.7).

1.2.1. The case of a p-adic field. Fix for this subsection a finite extension $L/\mathbf{Q}_p$. We normalize the p-adic valuation on L so that $|p| = 1/p$.

We fix rational numbers $0 < s \leq r$. The first space we define is the p-adic annulus

$$\mathbf{A}^1[s,r] = \left\{ T \colon p^{-r/(p-1)} \leq |T| \leq p^{-s/(p-1)} \right\}.$$

Its associated affinoid algebra, which we denote by $\mathcal{R}^{[s,r]}$ (the letter $\mathcal{R}$ stands for "Robba"), is isomorphic to the Laurent domain (at least if $r < \infty$)

$$\mathbf{Q}_p\langle T, X, Y\rangle / (X - Tp^{s/(p-1)}, YTp^{r/(p-1)} - 1).$$

If we take $r = \infty$ then we get a p-adic disc

$$\mathbf{A}^1[s,\infty] = \left\{ T \colon \ |T| \leq p^{-s/(p-1)} \right\}.$$

The associated affinoid algebra is denoted by $\mathcal{R}^{[s,\infty]}$ and it is isomorphic to a Weierstrass domain

$$\mathbf{Q}_p\langle T, X\rangle / (X - Tp^{s/(p-1)}).$$

Here is a picture of the disc so that we can can keep track of the way the radii are changing.

Fix r and notice that if $0 < s < s' \leq r$ then the restriction map defines an inclusion $\mathcal{R}^{[s,r]} \subset \mathcal{R}^{[s',r]}$ which is flat and has dense image (see [**41**, Remark 2.1.4]). We then define

$$\mathcal{R}^r = \bigcap_{0<s\leq r} \mathcal{R}^{[s,r]}.$$

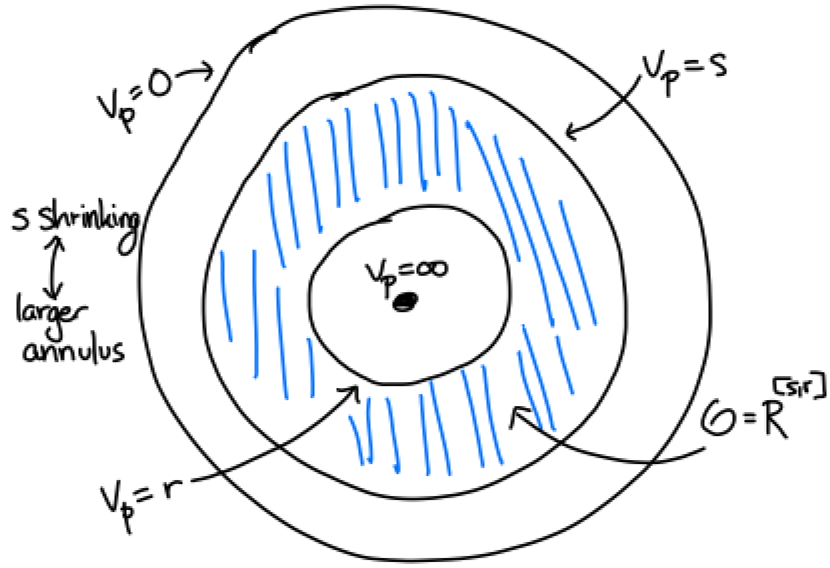
This is the ring of functions converging on a half open annulus

$$\mathbf{A}^1(0,r] = \left\{ T \colon p^{-r/(p-1)} \leq |T| < 1 \right\}.$$

inside the disc. This is no longer an affinoid because, for example, the function T on $\mathbf{A}^1(0,r]$ would violate the maximum modulus principle [**11**, Proposition 6.2.1/4]. However, it is a rigid analytic space in the sense of Tate, admissibly covered by admissible open subsets $\left\{\mathbf{A}^1[s,r]\right\}$ with $0 < s \leq r$.

DEFINITION. *The Robba ring $\mathcal{R}$ is the ring $\mathcal{R} = \bigcup_{0<r<\infty} \mathcal{R}^r$.*

Figure 1. A picture of the disc



Suppose that L is a p-adic field. We extend the definition to L-coefficients by the formula $\mathcal{R}_L^? = \mathcal{R}^? \otimes_{\mathbf{Q}_p} L$ where ? is either of the decorations r or $[s,r]$. It would be the same to begin with spaces $\mathbf{A}^1[s,r]_{/L}$ with ring of functions of $L\langle T, X, Y\rangle/(X - Tp^{s/(p-1)}, YTp^{r/(p-1)} - 1)$. In particular we still have

$$\mathcal{R}_L^r = \bigcap_{0<s\leq r} \mathcal{R}_L^{[s,r]} \quad \text{and} \quad \mathcal{R}_L = \bigcup_{0<r<\infty} \mathcal{R}_L^r.$$

Now, every element $f \in \mathcal{R}_L$ is given by a formal Laurent series

$$f(T) = \sum_{n\in\mathbf{Z}} a_n T^n, \qquad a_n \in L$$

so that the sum converges on some annulus, depending on f.

1.2.2. The case of a general affinoid base. We extend the Robba ring to the situation of an affinoid base. Throughout this section we will denote by A an affinoid $\mathbf{Q}_p$-algebra (recall §1.1) and $X = \mathrm{Sp}(A)$. Let $0 < s \leq r$ be rational numbers again. Then, define spaces

$$X^{[s,r]} = X \times_{\mathbf{Q}_p} \mathbf{A}^1[s,r]$$
$$X^r = X \times_{\mathbf{Q}_p} \mathbf{A}^1(0,r]$$

with rings of functions $\mathcal{R}_X^{[s,r]} = A\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{R}^{[s,r]}$ and $\mathcal{R}_X^r = A\widehat{\otimes}_{\mathbf{Q}_p}\mathcal{R}^r$. Notice that $X^{[s,r]}$ is an affinoid space, i.e. $\mathcal{R}_X^{[s,r]}$ is an affinoid $\mathbf{Q}_p$-algebra. In particular, it is noetherian. Note that we have as well that

$$\mathcal{R}_X^r = \bigcap_{0<s\leq r} \mathcal{R}_X^{[s,r]}.$$

Definition. *The Robba ring over X (or, over A) is $\mathcal{R}_X = \bigcup_{0<r<\infty} \mathcal{R}_X^r$.*

Example 1.5. In the case that A is a finite $\mathbf{Q}_p$-algebra. Then, we will often switch to the notation $\mathcal{R}_A^? := \mathcal{R}_{\mathrm{Sp}(A)}^?$ where $? \in \{\varnothing, [s,r], r\}$. Notice that in this case, since A is finite-dimensional over $\mathbf{Q}_p$, we have

$$\mathcal{R}_A^? = A \otimes_{\mathbf{Q}_p} \mathcal{R}^?.$$

For example, this notation agrees with the previous notation for $\mathcal{R}_L$ where $L/\mathbf{Q}_p$ is a p-adic field.

EXAMPLE 1.6. If A is a local Artin $\mathbf{Q}_p$-algebra then $A/\mathfrak{m}_A$ is a p-adic field and one sees that $\mathcal{R}_A$ is a successive extension of $\mathcal{R}_{A/\mathfrak{m}_A}$ by itself.

1.2.3. Modules over the Robba ring. We now review and create some basic results in the theory of modules over the Robba ring. What we are going to explain can be found in [**41**, §2] except in the next section we have to extend one small proof; thus, we feel it necessary to review definitions and details for a short time. We first consider modules over the Robba ring $\mathcal{R}_L$.

Since $\mathcal{R}_L^{[s,r]}$ is the ring of analytic functions on an affinoid subdomain of the unit disc, it is a noetherian integral domain. It follows that $\mathcal{R}_L^r$ and $\mathcal{R}_L$ are also an integral domains. The finiteness properties of $\mathcal{R}_L$ were first studied by Lazard [**45**]; he proved that any finitely generated ideal of $\mathcal{R}_L$ is principal. The following strengthening of Lazard's work is indispensable for us.

PROPOSITION 1.7 ([**8**, Proposition 4.12]). *The Robba ring $\mathcal{R}_L$ over L is an adequate Bezout domain. That means:*

(a) For a finite module M over $\mathcal{R}_L$ to be free, it is necessary and sufficient that it be torsion free.
(b) Any finitely generated submodule M of $\mathcal{R}_L^n$ has elementary divisors: there is a basis $e_1, \dots, e_n$ for $\mathcal{R}_L^n$ and elements $f_1 \mid \cdots \mid f_u$ of $\mathcal{R}_L$ such that $f_1e_1, \dots, f_ue_u$ is a basis for M (it is free by the previous part). Moreover, the ideals $(f_1) \supset (f_2) \supset \cdots \supset (f_u)$ are unique.

Fix now a rational number $r_0 > 0$. In general, we notice that $\left\{\mathbf{A}^1[s,r]\right\}_{0<s\leq r\leq r_0}$ is an admissible covering of $\mathbf{A}^1(0,r_0]$ by admissible affinoid open sets. More generally X^{r_0} is admissibly covered by $\left\{X^{[s,r]}\right\}$. Each $X^{[s,r]}$ is affinoid because X is affinoid. In particular, we have an equivalence of categories [**11**, 9.4.3/2]

$$\left\{\text{finite } \mathcal{R}_X^{[s,r]}\text{-modules}\right\} \leftrightarrow \left\{\text{coherent sheaves on } X^{[s,r]}\right\}.$$

Thus, to give a coherent sheaf $\mathcal{Q}$ on X^{r_0} is the same as to give:

(a) The data of a finite $\mathcal{R}_X^{[s,r]}$-module $Q^{[s,r]}$ for all $0 < s \leq r \leq r_0$.
(b) For all choices of intervals $[s',r'] \subset [s,r] \subset (0,r_0]$ we need isomorphisms

$$\mathrm{res}^{[s,r]}_{[s',r']} : Q^{[s,r]} \otimes_{\mathcal{R}_X^{[s,r]}} \mathcal{R}_X^{[s',r']} \xrightarrow{\cong} Q^{[s',r']},$$

such that
(c) if we have three intervals $[s'',r''] \subset [s',r'] \subset [s,r] \subset (0,r_0]$ then we have compatibility

$$\begin{array}{ccc} Q^{[s'',r'']} & \xleftarrow{\mathrm{res}^{[s',r']}_{[s,r]}} & Q^{[s',r']} \otimes_{\mathcal{R}_X^{[s',r']}} \mathcal{R}_X^{[s'',r'']} \\ \uparrow {\scriptstyle \mathrm{res}^{[s',r']}_{[s'',r'']}} & & \uparrow {\scriptstyle \mathrm{res}^{[s,r]}_{[s',r']} \otimes \mathrm{id}} \\ Q^{[s,r]} \otimes_{\mathcal{R}_X^{[s,r]}} \mathcal{R}_X^{[s'',r'']} & \xrightarrow[(\mathrm{id}\otimes 1)\otimes \mathrm{id}]{} & \left(Q^{[s,r]} \otimes_{\mathcal{R}_X^{[s,r]}} \mathcal{R}_X^{[s',r']}\right) \otimes_{\mathcal{R}_X^{[s',r']}} \mathcal{R}_X^{[s'',r'']} \end{array}$$

We will use the notation $\mathcal{Q} = (Q^{[s,r]})$ to pass between the sheaf on X^{r_0} and the modules over each member $X^{[s,r]}$ of the cover.

Notice that the data of (b) gives us that if $0 < s \leq s' \leq r_0$ then we have a canonical map $Q^{[s,r_0]} \to Q^{[s',r_0]}$ and the module of global sections is

$$Q := \Gamma(X^{r_0}, \mathcal{Q}) = \varprojlim{}_{s \to 0^+} Q^{[s,r_0]}.$$

This is a module over $\mathcal{R}_X^{r_0}$. However, one cannot expect in general to have Q inherit any finiteness properties as a module over $\mathcal{R}_X^{r_0}$. To explain the precise situation under which we can expect this, we make the following definiton.

DEFINITION. *A coherent sheaf $\mathcal{Q} = (Q^{[s,r]})$ on X^{r_0} is said to be uniformly finitely presented if there exists a countable cover $\{[s_i, r_i]\}_{i \in \mathbf{N}}$ for $(0, r_0]$ and a pair of integers $(m, n) \in \mathbf{N}^{\oplus 2}$ such that each module $Q^{[s_i,r_i]}$ has a finite presentation*

$$\left(\mathcal{R}_X^{[s_i,r_i]}\right)^{\oplus m} \to \left(\mathcal{R}_X^{[s_i,r_i]}\right)^{\oplus n} \to Q^{[s_i,r_i]} \to 0.$$

We say that $\mathcal{Q}$ is uniformly finite projective if in addition each $Q^{[s,r]}$ projective (or, what is the same, flat).

Note that if we fix $s < r$ then to check $Q^{[s,r]}$ is flat it suffices to check that $Q^{[s',r']}$ is flat for $[s', r'] \supset [s, r]$. Thus, to check that every module $Q^{[s,r]}$ in a coherent sheaf $\mathcal{Q}$ is flat, it is enough to see it on a countable cover by the data (b).

PROPOSITION 1.8. *The natural functor*

$$\left\{ \begin{matrix} \text{finitely presented} \\ \mathcal{R}_X^{r_0}\text{-modules} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{uniformly finitely presented} \\ \text{coherent sheaves on } X^{r_0} \end{matrix} \right\}$$

$$Q \mapsto (Q \otimes_{\mathcal{R}_X^{r_0}} \mathcal{R}_X^{[s,r]})_{0 < s \leq r \leq r_0}$$

is an equivalence of categories. Moreover, it induces an equivalence between the subcategories

$$\left\{ \begin{matrix} \text{finite projective} \\ \mathcal{R}_X^{r_0}\text{-modules} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{uniformly finite projective} \\ \text{coherent sheaves on } X^{r_0} \end{matrix} \right\}.$$

PROOF. That $(Q^{[s,r]}) = (Q \otimes_{\mathcal{R}_X^{r_0}} \mathcal{R}_X^{[s,r]})$ is a uniformly finitely presented coherent sheaf is clear since any finite presentation of Q over $\mathcal{R}_X^{r_0}$ provides us with a uniform finite presentation for the $Q^{[s,r]}$. To see that it is essentially surjective we note that if $\mathcal{Q} = (Q^{[s,r]})$ is a coherent sheaf then

(a) $Q^{[s,r]} \cong \Gamma(X^{r_0}, \mathcal{Q}) \otimes_{\mathcal{R}_X^{r_0}} \mathcal{R}_X^{[s,r]}$ by [**41**, Lemma 2.1.6(3)], and
(b) if $\mathcal{Q}$ is uniformly finitely presented (resp. uniformly finite projective) then $\Gamma(X^{r_0}, \mathcal{Q})$ is finitely presented over $\mathcal{R}_X^{r_0}$ (resp. finite projective) by Proposition 2.1.15 of *loc. cit.*.

This completes the proof. ☐

With this in hand, we will now deal exclusively with the global sections of coherent sheaves. However, notice that the Proposition 1.3 and Corollary 1.4 apply to the global sections of any coherent sheaf now.

REMARK. The rings $\mathcal{R}_X^{[s,r]}$ are all noetherian and thus the category of finitely presented modules over $\mathcal{R}_X^{[s,r]}$ is abelian. The ring $\mathcal{R}$ is not noetherian, but the category of finitely presented modules over $\mathcal{R}$ is abelian still by Proposition 1.7. Is the same true for $\mathcal{R}_X$ (or $\mathcal{R}_X^{r_0}$)? Note that by [**41**, Lemma 2.1.12, Proposition 2.1.15(1)], if $\alpha : Q \to Q'$ is a surjection of finitely presented $\mathcal{R}_X^{r_0}$ modules then $\ker(\alpha)$ is finitely generated over $\mathcal{R}_X^{r_0}$.

1.3. (φ, Γ)-modules

Proposition 1.8 tell us that in order to obtain a finite projective $\mathcal{R}_X^{r_0}$-module we will need to construct a compatible system of finite $\mathcal{R}_X^{[s,r]}$-modules whose number of generators does not expand as s and r varies, i.e. the generators can be spread out over the entire half open annulus. One situation where this arises is case of φ-modules so we begin with that theory. We end the section with a brief discussion of the relation between (φ, Γ)-modules and Galois representations.

1.3.1. Definition. On the ring $\mathcal{R}$ there is a collection of $\mathbf{Q}_p$-linear operators. Denote by Γ the group $\mathbf{Z}_p^\times$. Then, for each $\gamma \in \Gamma$ we define an operator on $\mathcal{R}$ by the formulas

$$(\gamma \cdot f)(T) = f((1+T)^\gamma - 1).$$

We have as well one more operator

$$(\varphi \cdot f)(T) = f((1+T)^p - 1).$$

It can be checked that these operations are well-defined (i.e. preserve the convergence condition on $\mathcal{R}$) and that they commute. Notice that if we denote by $r(f)$ the radius on which f converges then $r(\gamma f) = r(f)$ and $r(\varphi f) \leq r(f)$ (and often $r(\varphi f) = r(f)/p$). In fact, φ induces natural inclusions

$$\varphi : \mathcal{R}^{[s,r]} \hookrightarrow \mathcal{R}^{[s/p,r/p]},$$
$$\varphi : \mathcal{R}^{r_0} \hookrightarrow \mathcal{R}^{r_0/p}$$

and presents the larger rings as finite free modules of rank p over the smaller rings. If we have a module Q over $\mathcal{R}^{r_0}$ we thus have two different ways of extending scalars from $\mathcal{R}^{r_0}$ to $\mathcal{R}^{r_0/p}$. We denote by φ^*Q the tensor product $Q \otimes_{\mathcal{R}^{r_0}} \mathcal{R}^{r_0/p}$ where we use $\varphi : \mathcal{R}^{r_0} \to \mathcal{R}^{r_0/p}$ and $Q\big|_{(0,r_0/p]}$ the same tensor product but by using the restriction map $\mathrm{res}^{r_0}_{r_0/p} : \mathcal{R}^{r_0} \to \mathcal{R}^{r_0/p}$. If $X = \mathrm{Sp}(A)$ we extend the definition(s) to $\mathcal{R}_X^?$ by making φ and Γ act trivially on the coefficients A. We begin by studying the action of φ on modules over $\mathcal{R}_X^{r_0}$.

Definition. *A generalized φ-module over $\mathcal{R}_X^{r_0}$ is a finitely presented $\mathcal{R}_X^{r_0}$-module Q together with an isomorphism $\varphi^*Q \cong Q\big|_{(0,r_0/p]}$ of $\mathcal{R}_X^{r_0/p}$-modules . We say that Q is a φ-module (dropping the word generalized) if Q is projective.*

If $\alpha : \varphi^*Q \to Q\big|_{(0,r_0/p]}$ is a given isomorphism then we obtain an operator

$$Q \hookrightarrow \varphi^*Q \xrightarrow{\alpha} Q\big|_{(0,r_0/p]}$$

which we denote by φ as well. Moreover, if $f \in \mathcal{R}_X^{r_0}$ then $\varphi(fq) = \varphi(f)\varphi(q)$ (these are all elements of $Q\big|_{(0,r_0/p]}$).

If Q is a generalized φ-module then by Proposition 1.8 we know that it arises as the global sections of a coherent sheaf $\mathcal{Q} = (Q^{[s,r]})$ on X^{r_0}. Moreover, it is clear to see that the condition $\varphi^*Q \cong Q\big|_{(0,r_0/p]}$ is the same as the the condition that there are isomorphisms $\varphi^*Q^{[s,r]} \cong Q^{[s/p,r/p]}$ as modules over $\mathcal{R}_X^{[s/p,r/p]}$, compatible with the restriction maps. In fact, the equivalence of Proposition 1.8 gives us

Proposition 1.9. *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{generalized } \varphi\textit{-modules} \\ \textit{over } \mathcal{R}_X^{r_0} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{coherent sheaves } \mathcal{Q} = (Q^{[s,r]}) \textit{ on } X^{r_0} \textit{ such that} \\ \varphi^*Q^{[s,r]} \cong Q^{[s/p,r/p]} \textit{ compatible with restrictions} \end{array} \right\}$$

Proof. Indeed, you just need to check that any element of the left hand side arises from the right hand side, i.e. such a $\mathcal{Q}$ is automatically uniformly finitely presented. However, choosing a finite presentation for $Q^{[r_0/p,r_0]}$ and using the condition that $(\varphi^*)^n Q^{[r_0/p,r_0]} \cong Q^{[r_0/p^{n+1},r_0/p^n]}$ we see that this is the case. $\square$

By [**11**, Proposition 9.4.3/2], the category of coherent sheaves on X^{r_0} is abelian. The extra condition on the right hand side is such that we can deduce the following corollary.

Corollary 1.10. *The category of generalized φ-modules is abelian.*

Proof. We just need to check that if $\mathcal{Q} \to \mathcal{Q}'$ is a morphism of coherent sheaves on the right hand side of Proposition 1.9 then $\operatorname{coker}(f)$ and $\ker(f)$ both satisfy the extra compatibilities. Since φ presents $\mathcal{R}_X^{[r/p,s/p]}$ as a free $\mathcal{R}_X^{[r,s]}$-module we have that $\varphi^*(-)$ is exact. The result follows from this. For example,

$$\varphi^* \operatorname{coker}\left(Q^{[r,s]} \xrightarrow{f} Q'^{[r,s]}\right) \cong \operatorname{coker}\left(\varphi^* Q^{[r,s]} \to \varphi^* Q'^{[r,s]}\right).$$

$\square$

We now define the main objects which we will be dealing with. Recall that $\Gamma = \mathbf{Z}_p^\times$.

Definition. *A generalized (φ,Γ)-module Q over $\mathcal{R}_X^{r_0}$ (or, over X^{r_0}) is a generalized φ-module over $\mathcal{R}_X^{r_0}$ together with a continuous $\mathcal{R}_X^{r_0}$-semi-linear action of Γ commuting with φ.*

By a generalized (φ,Γ)-module over $\mathcal{R}_X$ (or, over X) we mean an $\mathcal{R}_X$-module $Q = Q^{r_0} \otimes_{\mathcal{R}_X^{r_0}} \mathcal{R}_X$ where Q^{r_0} is a (φ,Γ)-module over $\mathcal{R}_X^{r_0}$.

If we drop the word generalized throughout, i.e. work with projective objects, we get (φ,Γ)-modules.

Continuing the discussion above, to give a generalized φ-module over $\mathcal{R}_X$ is the same as to give an $\mathcal{R}_X$-semi-linear operator φ on Q such that $\varphi(Q)$ generates Q over $\mathcal{R}_X$. To give a (φ,Γ)-module, we add in the data of a semi-linear operation of the group Γ. In fact, this is often how we will specify a (φ,Γ)-module. Notice that if Q is a generalized (φ,Γ)-module then the φ-condition becomes (the possibly more familiar) condition that $\varphi^*Q \cong Q$. By a map $f : Q \to Q'$ in the category of generalized (φ,Γ)-modules over X is a continuous, (φ,Γ)-equivariant $\mathcal{R}_X$-map. By definition, such a map arises as a map $f^{r_0} : Q^{r_0} \to Q'^{r_0}$ for a sufficiently large r_0. When we want to emphasize the equivariance, we might use a subscript (φ,Γ). For example, we write

$$\operatorname{Hom}_{(\varphi,\Gamma)}(Q,Q')$$

to be the $\mathcal{R}_X$-module of equivariant morphisms $Q \to Q'$. This is again a generalized (φ,Γ)-module. If Q is a (φ,Γ)-module and we let $Q' := \mathcal{R}_X$ then we get the dual (φ,Γ)-module $Q^\vee$.

We finish with a lemma which will be used in Chapter 4. It is meant to illustrate how we can always descend from modules over $\mathcal{R}_X$ to a question about finite modules over the noetherian affinoid spaces $X^{[s,r]}$.

Lemma 1.11. *Assume that X is reduced. Let Q be a generalized (φ,Γ)-module and $f : \mathcal{R}_X \to Q$ a (φ,Γ)-equivariant map. Suppose that there is a Zariski dense set $U \subset X$ such that the base change $f_u : \mathcal{R}_{L(u)} \to Q_u$ is injective. Then, f is injective.*

Proof. Choose an $r_0 > 0$ such that Q arises from base change from $\mathcal{R}_X^{r_0}$ and f arises from base change by a map $f^{r_0} : \mathcal{R}_X^{r_0} \to Q^{r_0}$. It suffices to show that f^{r_0} is injective. If $0 < s \leq r_0$ then we have the induced map

$$f^{[s,r_0]} : \mathcal{R}_X^{[s,r_0]} \to Q^{[s,r_0]} = Q^{r_0} \otimes_{\mathcal{R}_X^{r_0}} \mathcal{R}_X^{[s,r_0]}. \tag{1.2}$$

By Lemma [**41**, Lemma 2.1.6(2)] it suffices to show that each $f^{[s,r_0]}$ is injective. Indeed, we have $\ker(f^{r_0}) = \varprojlim \ker(f^{[s,r_0]})$ and $\varprojlim$ is exact by *loc. cit.* as well.

To prove that $f^{[s,r_0]}$ is injective, we use that $\mathcal{R}_X^{[s,r_0]}$ is noetherian. Let $I := \mathrm{im}(f^{[s,r_0]})$. If $u \in U$ we know that f_u is injective since its composition with $I \otimes_X L(u) \to Q_u$ is. By definition f_u is also surjective and thus an isomorphism. In particular, for all $v \in U^{[s,r_0]}$ we have that $\dim_{L(y)} I_y = 1$ (notice y is a point in the product $U^{[s,r_0]}$ now). Since $X^{[s,r_0]}$ is affinoid, $U^{[s,r_0]} \subset X^{[s,r_0]}$ is Zariski dense and everything in sight is finitely generated we deduce by Proposition 1.3 that $\dim_{L(y)} I_y \geq 1$ for all $y \in X^{[s,r_0]}$. On the other hand, the upper bound $\dim_{L(v)} I_y \leq 1$ is tautological since I is a quotient of $\mathcal{R}_X^{[s,r_0]}$. Since $X^{[s,r_0]}$ is reduced and $y \mapsto \dim_{L(y)} I_y$ is constant, we get that I is flat over $X^{[s,r_0]}$. Thus, it follows that $\mathcal{R}_X^{[s,r_0]} \cong I$ and we are done. □

Remark. Suppose that $t \in \mathcal{R}$ (no subscript!). We have the same result if we replace $\mathcal{R}_X$ by $\mathcal{R}_X/t$. Indeed, all we used is that we arose from base change from one of the noetherian spaces which build up $\mathcal{R}_X$. In the case of $\mathcal{R}_X/t$ we would work with $X \times_{\mathbf{Q}_p} Z(t)$ where $Z(t) \subset \mathbf{A}^1(0, r_0]$ is the zero locus of the function t, for r_0 sufficiently large.

1.3.2. Relationship with Galois representations. We now recall the relationship between Galois representations and (φ,Γ)-modules. If A is an affinoid $\mathbf{Q}_p$-algebra then by a representation of $G_{\mathbf{Q}_p}$ over A we mean a finite projective A-module V equipped with a continuous A-linear action of the local Galois group $G_{\mathbf{Q}_p}$.

Proposition 1.12. *Let $A/\mathbf{Q}_p$ be a finite $\mathbf{Q}_p$-algebra. There is a fully faithful $\otimes$-equivalent embedding of categories*

$$D_{\mathrm{rig}} : \left\{A\text{-linear representations } V \text{ of } G_{\mathbf{Q}_p}\right\} \to \{(\varphi,\Gamma)\text{-modules over } \mathcal{R}_A\}.$$

It defines an equivalence of categories between the Galois representations and the so-called étale (φ,Γ)-modules. Moreover, D_{rig} induces an isomorphism

$$\mathrm{Ext}^1_{L[G_{\mathbf{Q}_p}]}(V,V) \cong \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_{\mathrm{rig}}(V), D_{\mathrm{rig}}(V)).$$

Let us make some remarks on the proof and its history. First, one can reduce to the case where A is a field by looking at the finitely many localizations $A_{\mathfrak{m}}$, each of which is a local Artin $\mathbf{Q}_p$-algebra and then proceeding by induction on the length. So, it remains to consider the case where L is a field. In that case, analogous results (see [**27**] and [**19**]) were first proved with the Robba ring replaced by different rings of analytic functions on affinoid subdomains of the disc. The key step in extending this previous work to the Robba ring is Kedlaya's theorem on slope filtrations [**42**, Theorem 6.10]. Kedlaya's theorem is also how one deduces the second point of the proposition (see [**5**, Proposition 2.2.6]).

Now let $X = \mathrm{Sp}(A)$ be the rigid space associated to an affinoid $\mathbf{Q}_p$-algebra. Looking ahead to the applications in Chapter 5 we see that we are going to *begin* with Galois representations (of a global Galois group) over an affinoid base like X and then study the behavior at p via (φ,Γ)-modules. Thus, it is important that one can convert a Galois representation over an affinoid base into a (φ,Γ)-module and vice versa, at least locally.

Proposition 1.13 ([**40**, Theorem 3.11, Theorem 0.2]). *There is a fully faithful embedding*

$$D_{\mathrm{rig}} : \left\{A\text{-linear representations } V \text{ of } G_{\mathbf{Q}_p}\right\} \hookrightarrow \{(\varphi,\Gamma)\text{-modules over } \mathcal{R}_X\}.$$

Moreover, if Q is a (φ,Γ)-module over $\mathcal{R}_X$ and $Q_x = D_{\mathrm{rig}}(V_x)$ for some point $x \in X$ and some Galois representation V_x over $L(x)$ then there exists an affinoid neighborhood U of X and an A-linear representation V_U over U such that $D_{\mathrm{rig}}(V_U) = Q\big|_U$ as (φ,Γ)-modules over $\mathcal{R}_U$.

Remark. We have stated the most precise result that we know, but we are actually only going to require that we can convert (φ, Γ)-modules back into Galois representations over finite thickenings of the residue fields (cf. §5.3). Thus, we could replace the "moreover" in the above statement with the "moreover" of Proposition 1.12 would be enough.

CHAPTER 2

Arithmetic theory of Galois representations and (φ, Γ)-modules

In this chapter we will expand on the theory of (φ, Γ)-modules over finite $\mathbf{Q}_p$-algebras. Recall that if L is a p-adic field then the category of continuous L-linear representations of $G_{\mathbf{Q}_p}$ is naturally a full subcategory of the category of (φ, Γ)-modules over $\mathcal{R}_L$. Our main goal is to explain how to fruitfully use this embedding to gain insight into the structure of Galois representations. Perhaps the most important, recent, development in understanding Galois representations through (φ, Γ)-modules has been Colmez's notion of *triangulations*. The definition is so simple we can give it in this introduction: a triangulation of a Galois representation V is a full (φ, Γ)-stable filtration (with free quotients) inside the (φ, Γ)-module $D_{\mathrm{rig}}(V)$. Furthermore, the construction of triangulations is simple in a very important situation: the set of triangulations of a crystalline Galois representation are in one-to-one correspondence with orderings of the crystalline eigenvalues, at least in a generic situation.

As it turns out, there are many more examples of trianguline representations. In fact, we will explain in Chapter 4 that certain p-adic families of Galois representations are canonically triangulated (though this particular fact was known, in some sense, prior to our work). This multitude of examples provides us with a natural setting in which to see classical Galois representations. In order to paint this beautiful picture, however, we need to set straight certain foundational aspects. Thus, our first goal in this chapter is to survey the arithmetic aspects of (φ, Γ)-modules.

To any given triangulation (of, say, a crystalline Galois representation) there is the notion of being *non-critical* or *critical*. It turns out that over a p-adic family, the canoncial triangulations mentioned above only suffice at over the non-critical locus. This realization forces us to consider a generalization of triangulations, known as parabolizations—the terminology is due to Chenevier. Our second main goal in this chapter is to draw out the notion of criticality from the theory of triangulations and justify the construction of parabolizations instead.

The arrangement is as follows. First, we expand on and precise the relationship between Galois representations and (φ, Γ)-modules which we began in the previous section. This includes a brief, but explicit, description of the extension of Fontaine's functors of p-adic Hodge theory from the category of Galois representations to the category of (φ, Γ)-modules. In the same spirit, we next review Herr's extension of the Galois cohomology. This includes the fundamental calculations of Colmez and Liu in the rank one case or, what amounts to the same, the trianguline case. It is in the third section that we carefully expose the fundamental notion of parabolizations and criticality.

2.1. Foundations of (φ, Γ)-modules

Throughout this section A will denote a finite $\mathbf{Q}_p$-algebra and in the case that A is a field we will write $A = L$. Recall Proposition 1.12 implies that $V \mapsto D_{\mathrm{rig}}(V)$ is a fully faithful embedding of the category of A-linear representations V of $G_{\mathbf{Q}_p}$ into the category of (φ, Γ)-modules over the Robba ring $\mathcal{R}_A$. We begin with just a brief tour of this relationship. One should also look ahead to the study of crystalline representations beginning with Proposition 2.21.

2.1.1. Examples. We first begin with the rank one case. Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be a continuous character. We define a rank one (φ, Γ)-module $\mathcal{R}_A(\delta)$ as follows. It is a free rank one $\mathcal{R}_A$-module with basis $\mathbf{e}$ and (φ, Γ)-action on $\mathbf{e}$ given by

$$\varphi\mathbf{e} = \delta(p)\mathbf{e} \qquad \gamma\mathbf{e} = \delta(\gamma)\mathbf{e}.$$

We extend the action to $\mathcal{R}_A\mathbf{e}$ semi-linearly with respect to the coefficients $\mathcal{R}_A$. The fact that δ is continuous implies that the action extends to all of Γ on $\mathcal{R}_A\mathbf{e}$. It arises from a module over $\mathcal{R}_A^{r_0}$ for any r_0.

On the other hand, we could first begin with a continuous character $\chi : G_{\mathbf{Q}_p} \to A^\times$. If $W_{\mathbf{Q}_p} \subset G_{\mathbf{Q}_p}$ is the Weil group of $\mathbf{Q}_p$ then local class field theory gives us a diagram

$$\begin{array}{ccccc} W_{\mathbf{Q}_p} & \longrightarrow & W_{\mathbf{Q}_p}^{\mathrm{ab}} & \xrightarrow{\mathrm{Art}_p^{-1}} & \mathbf{Q}_p^\times \\ \downarrow & & \downarrow & & \downarrow \\ G_{\mathbf{Q}_p} & \longrightarrow & G_{\mathbf{Q}_p}^{\mathrm{ab}} & \xrightarrow{\mathrm{Art}_p^{-1}} & \widehat{\mathbf{Q}_p}^\times \end{array}$$

where the local Artin maps Art_p are isomorphisms. The character χ factors through $G_{\mathbf{Q}_p}^{\mathrm{ab}}$ and thus we can consider the character

$$\delta_\chi := \chi|_{W_{\mathbf{Q}_p}^{\mathrm{ab}}} \circ \mathrm{Art}_p : \mathbf{Q}_p^\times \to A^\times.$$

Then, $D_{\mathrm{rig}}(\chi) = \mathcal{R}_A(\delta_\chi)$. The following completely describes *all* the rank one (φ, Γ)-modules.

Proposition 2.1 ([**5**, Proposition 2.3.1]). *If D is a (φ, Γ)-module of rank one over $\mathcal{R}_A$ then $D \cong \mathcal{R}_A(\delta)$ for some continuous character $\delta : \mathbf{Q}_p^\times \to A^\times$. Such a (φ, Γ)-module arises from a Galois character χ if and only if $v_p(\delta(p) \bmod \mathfrak{m}_A) = 0$.*

We now give some typical examples. Here, and forever, we denote $z : \mathbf{Q}_p^\times \to \mathbf{Q}_p^\times$ is the identity map and $|z| : \mathbf{Q}_p^\times \to \mathbf{Q}_p^\times$ is the p-adic norm.

Example 2.2. Let $V = \chi_{\mathrm{cycl}}$ be the the p-adic cyclotomic character. Since the local Artin map is normalized so that $p \mapsto \mathrm{Frob}_p$ we have that for $x \in \mathbf{Q}_p^\times$,

$$\left(\chi_{\mathrm{cycl}}|_{W_{\mathbf{Q}_p}^{\mathrm{ab}}} \circ \mathrm{Art}_p\right)(x) = xp^{-v_p(x)}.$$

In particular, $\delta_{\chi_{\mathrm{cycl}}}(p) = 1$ and $\delta_{\chi_{\mathrm{cycl}}}(\gamma) = \gamma$ for $\gamma \in \mathbf{Z}_p^\times$. We write this thusly: $\delta_{\chi_{\mathrm{cycl}}} = z\,|z|$. If n is an integer we can take the nth power of this map and get that $\delta_{\mathbf{Q}_p(n)}(p) = (z\,|z|)^n$.

Example 2.3. Let $\alpha \in L^\times$. Then we define a character $\mathrm{unr}(\alpha) : \mathbf{Q}_p^\times \to L^\times$ given by $p \mapsto \alpha$ and $\gamma \mapsto 1$ for $\gamma \in \mathbf{Z}_p^\times$. This arises as an unramified character χ of $G_{\mathbf{Q}_p}$ if and only if $v_p(\alpha) = 0$.

Example 2.4. If D is a (φ, Γ)-module over $\mathcal{R}_A$ and $\delta : \mathbf{Q}_p^\times \to A^\times$ is a continuous character then we denote

$$D(\delta) := D \otimes_{\mathcal{R}_A} \mathcal{R}_A(\delta).$$

We refer to this module as being obtained by twisting D by δ. Mimicking the situation with Galois representation, we use the notation $D(n)$ to denote $D\left((z\,|z|)^n\right)$, for $n \in \mathbf{Z}$.

Suppose still that $\delta : \mathbf{Q}_p^\times \to A^\times$ is a continuous character and we consider $\mathcal{R}_A(\delta)$. We define *the weight of* δ as

$$\mathrm{wt}(\delta) := -\frac{\log(\delta(1+p))}{\log(1+p)} = -\frac{d}{d\gamma}\Big|_{\gamma=1}\delta(\gamma).$$

Such weights will appear as the Hodge-Tate-Sen weights of trianguline (φ, Γ)-modules (cf. Example 2.9 and Lemma 2.20).

Example 2.5. For example, $\mathrm{wt}(z\,|z|) = -1$, which agrees with our normalization that the cyclotomic character χ_{cycl} has Hodge-Tate weight -1.

Example 2.6. Going back to the twisting for a moment, suppose that A is Artinian and $\kappa \in \mathfrak{m}_A \subset A$. Then, we can define the character $\kappa : \mathbf{Q}_p^\times \to A^\times$ given by $z \mapsto \exp(\kappa \log_p(z))$ (the series converges because κ is nilpotent). We have that the character κ has weight $-\kappa$. Thus if D is a (φ, Γ)-module over $\mathcal{R}_A$ we can form the twist $D(\kappa)$ and "change the weight".

One specific thing we have somehow managed not to mention is the element $t \in \mathcal{R}_L$. It plays the role of a period for the p-adic exponential function and permeates the entire theory. To that end, we define

$$t = \log(1+T) = T - \frac{T^2}{2} + \frac{T^3}{3} - \cdots .$$

This converges on $\mathbf{A}^1(0,\infty]$. Recall that we defined the action of φ and Γ in §1.3.1. If we compute the action on t we see easily that $\varphi(t) = pt$ and $\gamma(t) = \gamma t$. Thus $t \in \mathcal{R}_L$ is an eigenvector for the operators φ and Γ. Said another way, the submodule $t\mathcal{R}_L$ is a rank one (φ, Γ)-submodule. In the previous notation we have $t\mathcal{R}_L = \mathcal{R}_L(z)$. Similarly, $t^r\mathcal{R}_L$ is a rank one submodule of $\mathcal{R}_L$ where the action is through the character $z \mapsto z^r$. Its weight is $-r$. Notice that if $D \subset \mathcal{R}_L$ is any (φ, Γ)-submodule then by Proposition 1.7 we know that it is rank one, generated by some element $f \in \mathcal{R}_L$. The following tells us all the possible f and is a first step in the calculation of the cohomology groups (see Proposition 2.13).

Proposition 2.7 ([**23**, Remark 3.3]). *Every (φ, Γ)-submodule of $\mathcal{R}_L$ is of the form $t^r\mathcal{R}_L$ with $r \geq 0$.*

Recall, we have defined generalized (φ, Γ)-modules on page 24. If D is a generalized (φ, Γ)-module over $\mathcal{R}_L$ (which is a domain) then we denote by D_{tor} the $\mathcal{R}_L$-torsion submodule. The quotient D/D_{tor} is then a finitely generated, torsion-free $\mathcal{R}_L$-module. Following Proposition 1.7, we deduce that D/D_{tor} is free over $\mathcal{R}_L$ and the sequence

$$0 \to D_{\mathrm{tor}} \to D \to D/D_{\mathrm{tor}} \to 0$$

is split as $\mathcal{R}_L$-modules. In particular, D_{tor} is a generalized (φ, Γ)-module which is completely torsion. In fact, the following result tells us that every element in D_{tor} is killed by a power of t.

Corollary 2.8. *Let D be a generalized (φ, Γ)-module over $\mathcal{R}_L$. Then D is free if and only if D is t-torsion-free. If S is a torsion (φ, Γ)-module then there exists integers $m_1, \ldots, m_r \geq 1$ such that*

$$S \cong \bigoplus_{i=1}^{r} \mathcal{R}_L/t^{m_i}.$$

as $\mathcal{R}_L$-modules.

Proof. We use the notation as above. Our goal is to show that D_{tor} is t-torsion. Consider any finite presentation of $\mathcal{R}_L$-modules

$$0 \to D' \to \mathcal{R}_L^{\oplus d} \to D_{\mathrm{tor}} \to 0$$

with D' finitely generated over $\mathcal{R}_L$. Applying Proposition 1.7 again we see that there is a basis $e_1, \ldots, e_d$ for $\mathcal{R}_L^{\oplus d}$ and element $f_1 \mid \cdots \mid f_i$ such that D' is free on the elements $f_j e_j$. We claim that the submodule $f_j\mathcal{R}_L$ is (φ, Γ)-stable inside $\mathcal{R}_L$. Indeed, since Γ acts invertibly on D_{tor} and $\varphi^* D_{\mathrm{tor}} \cong D_{\mathrm{tor}}$ we have that for each γ, $\gamma f_1, \ldots, \gamma f_i$ is a list of elementary divisors for D_{tor} as well as $\varphi f_1, \ldots, \varphi f_i$. Since the ideals (f_j) are unique, we see that they are all (φ, Γ)-stable. Then, by the Proposition 2.7 they must all be of the form $t^{r_j}\mathcal{R}_L$. Thus D_{tor} is t-torsion. This also proves the final statement. □

Definition. *We say that D is a torsion (φ,Γ)-module if $D = D_{\mathrm{tor}}$. We say that D is pure torsion if D is free over $\mathcal{R}_L/t^r$ for some $r > 0$.*

If D is a free $\mathcal{R}_L$-module and $D' \subset D$ is a submodule then we can define the saturation of D' inside D as

$$D'^{\,\mathrm{sat}} = D \cap (D' \otimes_{\mathcal{R}_L} \mathrm{Frac}(\mathcal{R}_L)).$$

This is the largest submodule of D with the same rank as D' and whose quotient $D/D'^{\,\mathrm{sat}}$ is free over $\mathcal{R}_L$. By Corollary 2.8 we have that in the case that D is a (φ,Γ)-module and D' is a (φ,Γ)-submodule then

$$D'^{\,\mathrm{sat}} = D \cap D'[1/t].$$

For example, the saturation of $t^r\mathcal{R}_L$ inside $\mathcal{R}_L$ is just $\mathcal{R}_L$ itself.

2.1.2. p-adic Hodge theory of (φ,Γ)-modules. We now take a moment to recall the extension of Fontaine's p-adic Hodge theory [**28**] from the category $G_{\mathbf{Q}_p}$-representations to the category of (φ,Γ)-modules. The original reference for this material is [**8**]. Our recollection has been patched together from §5 of *loc. cit.*, [**5**, §2.2.7] and [**53**, §3]. We assume, however, that the reader is familiar with Fontaine's original theory (so that they can make sense of Proposition 2.10). Specifically, we freely use the notations $D_{\mathrm{Sen}}, D_{\mathrm{dR}}$, etc. for Fontaine's functors

$$\left\{L\text{-linear representations of } G_{\mathbf{Q}_p}\right\} \longrightarrow \{\text{some linear algebraic category}\}.$$

and hope that they make sense.

Throughout this discussion we work only over $\mathbf{Q}_p$. If we extend the coefficients to some p-adic field L then all it amounts to is changing statements below about the rank from $\mathcal{R}$ to $\mathcal{R}_L$ and dimensions to become over L. Let D be a (φ,Γ)-module over $\mathcal{R}$ and fix an r_0 such that D arises via base change from a (φ,Γ)-module D^{r_0} over $\mathcal{R}^{r_0}$. We begin first with Sen's theory (see [**57**] for the Galois representation story). For n sufficiently large, all of the p^nth roots of unity live inside the half open annulus $\mathbf{A}^1(0,r_0]$. For each choice ζ_{p^n} of such a root of unity we get a surjection $\mathcal{R}^{r_0} \twoheadrightarrow \mathbf{Q}_p(\zeta_{p^n})$ and thus we can define

$$D_{\mathrm{Sen}}(D) := (D^{r_0} \otimes_{\mathcal{R}^{r_0}} \mathbf{Q}_p(\zeta_{p^n})) \otimes_{\mathbf{Q}_p(\zeta_{p^n})} \mathbf{Q}_p(\zeta_{p^\infty}).$$

Here, $\mathbf{Q}_p(\zeta_{p^\infty}) = \varinjlim \mathbf{Q}_p(\zeta_{p^n})$ and we view Γ as the Galois group $\mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$ identified with $\mathbf{Z}_p^\times$ via the cyclotomic character

$$\mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p) = \Gamma \xrightarrow{\chi_{\mathrm{cycl}}} \mathbf{Z}_p^\times.$$

The object $D_{\mathrm{Sen}}(D)$ is a vector space over $\mathbf{Q}_p(\zeta_{p^\infty})$ of dimension equal to $\mathrm{rank}_{\mathcal{R}} D$. Further, it is equipped with a $\mathbf{Q}_p(\zeta_{p^\infty})$-semi-linear action of the group Γ. The formula we have given explicitly depends on r_0 and n, though for r_0 sufficiently small and n sufficiently large it is independent of such a choice. The $\mathbf{Q}_p(\mu_{p^\infty})$-linear operator

$$\Theta_{\mathrm{Sen}}(x) = -\frac{d}{d\gamma}\big|_{\gamma=1}(x)$$

is well-defined on $D_{\mathrm{Sen}}(D)$ and its characteristic polynomial has coefficients over $\mathbf{Q}_p$. The functor $D \mapsto D_{\mathrm{Sen}}(D)$ is exact because it is obviously right exact and preserves rank/dimension.

Definition. *The Hodge-Tate-Sen weights (with multiplicty) of D are the eigenvalues of Θ_{Sen} (with multiplicity) acting on $D_{\mathrm{Sen}}(D)$. We say that D is Hodge-Tate if its Hodge-Tate-Sen weights are all integers and the operator Θ_{Sen} is semi-simple. Finally, we say that D is regular if each Hodge-Tate-Sen weight has multiplicity one.*

EXAMPLE 2.9. If $D = \mathcal{R}_L(\delta)$ then it is easy to calculate that $\Theta_{\mathrm{Sen}} = \mathrm{wt}(\delta)$. The rank one (φ, Γ)-module $\mathcal{R}_L(\delta)$ is Hodge-Tate if and only if $\delta = z^r \varepsilon\, \mathrm{unr}(\alpha)$ where ε is a finite order character $\mathbf{Z}_p^\times \to L^\times$ and $\alpha \in L^\times$. In that case, its weight is $-r$.

Since t has a simple zero at each primitive p^nth root of unity, the completed local ring at any such point in X^{r_0} is a power series ring over t (with coefficients in the residue field). Thus for n sufficiently large we get a natural Γ-equivariant map

$$\iota_{r_0,n} : \mathcal{R}^{r_0} \to \mathbf{Q}_p(\zeta_{p^n})[[t]],$$

compatible with φ in the sense that the diagram

$$\begin{array}{ccc} \mathcal{R}^{r_0} & \xrightarrow{\iota_{r_0,n}} & \mathbf{Q}_p(\zeta_{p^n})[[t]] \\ \varphi \downarrow & & \downarrow t \mapsto pt \\ \mathcal{R}^{r_0/p} & \xrightarrow[\iota_{r_0/p,n+1}]{} & \mathbf{Q}_p(\zeta_{p^{n+1}})[[t]] \end{array} \tag{2.1}$$

is commuting. In any case, we use one of these maps to define

$$D^+_{\mathrm{dif}}(D) := (D^{r_0} \otimes_{\mathcal{R}^{r_0}} \mathbf{Q}_p(\zeta_{p^n})[[t]]) \otimes_{\mathbf{Q}_p(\zeta_{p^n})[[t]]} \mathbf{Q}_p(\zeta_{p^\infty})[[t]].$$

The compatibility (2.1) implies that the definition of D^+_{dif} it is independent of r_0 and n sufficiently large. We as well define

$$D_{\mathrm{dif}}(D) := D^+_{\mathrm{dif}}(D)[1/t] = D^{r_0} \otimes_{\mathcal{R}^{r_0}} \mathbf{Q}_p(\zeta_{p^\infty})((t)).$$

The functor $D \mapsto D^+_{\mathrm{dif}}(D)$ is again exact, as is $D \mapsto D_{\mathrm{dif}}(D)$ (being a localization of the other one). On $D_{\mathrm{dif}}(D)$ there is a $\mathbf{Q}_p(\zeta_{p^\infty})((t))$-semi-linear action of Γ again and we set

$$D_{\mathrm{dR}}(D) := D_{\mathrm{dif}}(D)^\Gamma.$$

Notice that $(\mathbf{Q}_p(\zeta_{p^\infty})((t)))^\Gamma = \mathbf{Q}_p$. Thus $D_{\mathrm{dR}}(D)$ is a $\mathbf{Q}_p$-vector space. One sees as well (as is usual in Fontaine's theory) from this that the natural map

$$D_{\mathrm{dR}}(D) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_{p^\infty})((t)) \to D_{\mathrm{dif}}(D)$$

is injective and thus

$$\dim_{\mathbf{Q}_p} D_{\mathrm{dR}}(D) \leq \mathrm{rank}_{\mathcal{R}}(D). \tag{2.2}$$

On the other hand, $(-)^\Gamma$ is only left exact in general and thus we can only expect D_{dR} to be left exact.

DEFINITION. *We say that D is de Rham if* $\dim_{\mathbf{Q}_p} D_{\mathrm{dR}}(D) = \mathrm{rank}_{\mathcal{R}} D$.

The $\mathbf{Q}_p$-vector space $D_{\mathrm{dR}}(D)$ has as well the extra structure of the Hodge filtration

$$\mathrm{Fil}^i D_{\mathrm{dR}}(D) := \left(D^{r_0} \otimes_{\mathcal{R}^{r_0}} t^i \mathbf{Q}_p(\zeta_{p^\infty})[[t]]\right)^\Gamma, \qquad i \in \mathbf{Z}.$$

We have $\mathrm{Fil}^{i+1} D_{\mathrm{dR}}(D) \subset \mathrm{Fil}^i D_{\mathrm{dR}}(D)$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(D) = D^+_{\mathrm{dif}}(D)^\Gamma$. This explains the "plus"-notation—it corresponds to the non-negative piece of the Hodge filtration. We also sometimes denote this by $D^+_{\mathrm{dR}}(D)$. Since $D_{\mathrm{dR}}(D)$ is finite-dimensional we see that the Hodge filtration is exhaustive (that is, $D_{\mathrm{dR}}(D) = \mathrm{Fil}^i D_{\mathrm{dR}}(D)$ for i small) and separated (that is, $(0) = \mathrm{Fil}^j D_{\mathrm{dR}}(D)$ for j large). Finally, we set

$$D_{\mathrm{cris}}(D) := (D[1/t])^\Gamma.$$

We evidently have an inclusion $D_{\mathrm{cris}}(D) \hookrightarrow D_{\mathrm{dR}}(D)$ (which depends on the choice of r_0 in the definitions above) and via this we can take the induced Hodge filtration $\mathrm{Fil}^i D_{\mathrm{cris}}(D)$ on $D_{\mathrm{cris}}(D)$. The operator φ on $D[1/t]$ preserves the Γ-invariants because φ and Γ commute and thus D_{cris} is a $\mathbf{Q}_p$-vectorspace together with a linear action of φ. Beware that these two actions of φ (on D versus $D_{\mathrm{cris}}(D)$) are related but by a dictionary one has to keep track of (cf. Proposition 2.21). Analogous to the de Rham case, the natural map

$$D_{\mathrm{cris}}(D) \otimes_L \mathcal{R}_L[1/t] \to D[1/t]$$

is injective. This implies that $\dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(D) \leq \mathrm{rank}_{\mathcal{R}} D$. Again, D_{cris} is not exact in general, but is left exact.

DEFINITION. *We say that D is crystalline if* $\dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(D) = \mathrm{rank}_{\mathcal{R}} D$.

In general we have that

$$\text{crystalline} \implies \text{de Rham} \implies \text{Hodge-Tate}.$$

Furthermore, for a de Rham representation we have that the Hodge-Tate-Sen weights (which are all integers) are the indexes where the Hodge filtration jumps and the multiplicity of a particular weight is read off by the dimension of the associated graded.

PROPOSITION 2.10. *Let V be an L-linear representation of $G_{\mathbf{Q}_p}$. Then there are canonical isomorphisms $\underline{D}(V) \cong \underline{D}(D_{\mathrm{rig}}(V))$ where $\underline{D}$ is any of the functors D_{Sen}, D_{dif}^+, D_{dR} or D_{cris} given above.*

PROOF. For D_{dif}^+ see [**8**, Corollarie 5.8] and for the others, see [**5**, Proposition 2.2.9]. ☐

We record the following for future use. Notice that since the quotient of two (φ, Γ)-modules need not be a (φ, Γ)-module (unlike Galois represenations), there is something to say at the beginning of the following proof.

LEMMA 2.11. *Suppose that D is de Rham (respectively, crystalline) and that $D' \subset D$ is a submodule. Then D' is de Rham (respectively, crystalline).*

PROOF. The proof is the same in either case so let us assume that D' is de Rham. If D'^{sat} is the saturation of D' inside D then by the remarks proceeding Corollary 2.8, and definition of D_{dR}, we have $D_{\mathrm{dR}}(D') = D_{\mathrm{dR}}(D'^{\mathrm{sat}})$. Thus, we may assume that D' is saturated inside D and the quotient D/D' is a (φ, Γ)-module. The argument now is a standard one from Fontaine's theory, using the left exactness of $D_{\mathrm{dR}}(-)$ and the inequality (2.2). ☐

We end this subsection with an explicit example which will elucidate the two different actions of φ which are floating around.

EXAMPLE 2.12. We consider the (φ, Γ)-module $t^n\mathcal{R}_L = \mathcal{R}_L(z^n) =: D_n$. This is Hodge-Tate of weight $-n$. Let $\mathbf{e} = t^n$ be the basis for D_n. Then the element $\mathbf{e}' = t^{-n}\mathbf{e}$ is fixed by Γ and φ acts on $\mathbf{e}'$ with eigenvalue 1. Thus D_n is crystalline and φ acts on $D_{\mathrm{cris}}(D_n)$ with the eigenvalue $\phi = 1$. The Hodge filtration is concentrated in degree $-n$.

In particular, we have that $D_{\mathrm{cris}}(D_n)$ is weakly admissible in the sense of Fontaine [**29**] if and only if $n = 0$. Since being weakly admissible is equivalent to being admissible that we recover that $t^n\mathcal{R}_L$ is the in image of D_{rig} if and only if $n = 0$. This agrees with Proposition 2.1.

On the other hand, consider $D = \mathcal{R}_L(z\,|z|)$ with a basis $\mathbf{e}$. This has Hodge-Tate weight -1 and we see that $t^{-1}\mathbf{e}$ is a basis for $D_{\mathrm{cris}}(D)$. Moreover,

$$\varphi(t^{-1}\mathbf{e}) = p^{-1}t^{-1}(p\,|p|)\mathbf{e} = p^{-1}(t^{-1}\mathbf{e}).$$

Thus φ acts on $D_{\mathrm{cris}}(D)$ with eigenvalue $\phi = p^{-1}$. Here now you see that $v_p(\phi)$ is the unique Hodge-Tate weight of D and thus $D_{\mathrm{cris}}(D)$ *does* arise from a Galois representation. Of course, $D = D_{\mathrm{rig}}(\chi_{\mathrm{cycl}})$.

Remark. The functor D_{cris} on Galois representations defines an equivalence of categories between all the crystalline Galois representations and the category of weakly admissible filtered φ-modules (over $\mathbf{Q}_p$). One of Berger's results [**9**] is that D_{cris} induces an equivalence of categories between the category of crystalline (φ, Γ)-modules and all the filtered φ-modules.

2.2. Galois cohomology I

In this section we continue of (φ, Γ)-modules by recalling the cohomology of (φ, Γ)-modules over a field. The computations in the rank one case will play a fundamental role in the construction of analytically varying filtrations of (φ, Γ)-modules over p-adic families in Chapter 4. In the final subsection we define and study the analog of the Bloch-Kato Selmer groups using the p-adic Hodge theory we just explained.

2.2.1. The Herr complex. Let L be a p-adic field. If V is an L-linear representation of $G_{\mathbf{Q}_p}$ then we use $H^i(G_{\mathbf{Q}_p}, V)$ to denote the continuous cohomology of $G_{\mathbf{Q}_p}$ with coefficients in V. These are finite-dimensional L-vector spaces concentrated in degrees $0 \leq i \leq 2$.

The extension of Galois cohomology from the category of L-linear representations of Galois groups to the category of (φ, Γ)-modules is due to Herr [**36**]. In the case of a generalized (φ, Γ)-module, Liu [**46**] defined and studied the cohomology. The extension to the torsion case, in particular, is used for deducing (φ, Γ)-versions of local Tate duality and the Euler-Poincaré-Tate characteristic formula (see §2.2.3). Some of the calculations we recall predate Liu's work and were carried out by Colmez [**23**] in his study of two-dimensional trianguline (φ, Γ)-modules.

Suppose that Q is a generalized (φ, Γ)-module over L. Denote by Δ the p-torsion subgroup of Γ (which only exists if $p = 2$). We choose a topological generator γ for Γ/Δ and we define two maps:

$$Q^{\Delta} \xrightarrow{d_\gamma^1} (Q^{\Delta})^{\oplus 2} \xrightarrow{d_\gamma^2} Q^{\Delta}$$

where

$$d_\gamma^1(x) = ((\varphi - 1)x, (\gamma - 1)x)\,,$$
$$d_\gamma^2((x, y)) = (\gamma - 1)x - (\varphi - 1)y.$$

It is easy to see that $d_\gamma^2 \circ d_\gamma^1 = 0$ and we denote by $C_\gamma^\bullet(Q)$ the associated complex. If γ and γ' are two different choices of generator Γ/Δ then $C_\gamma^\bullet(Q)$ is the same as $C_{\gamma'}^\bullet(Q)$ up to quasi-isomorphism, and so we define

$$H^i(Q) := H^i(C_\gamma^\bullet(Q)),$$

for any choice of γ. We have that $H^0(Q)$ is the set of simultaneous invariants $Q^{\varphi=1,\Gamma=1}$ for φ and Γ acting on Q. Alternatively,

$$H^0(Q) = \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L, Q).$$

As remarked in [**23**, §2.1] we have that if Q is free then $H^1(Q)$ is canonically isomorphic to $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_L, D)$. The cohomology is concentrated in degrees at most two, and we will see that the degree two term is related to the degree zero term (of a different module, see Proposition 2.16). In general these are finite-dimensional L-vector spaces and associated to a short exact sequence

$$0 \to Q' \to Q \to Q'' \to 0$$

of generalized (φ, Γ)-modules over L we get a long exact sequence

$$0 \to H^0(Q') \to H^0(Q) \to H^0(Q'') \to H^1(Q') \to H^1(Q) \to \cdots$$

in cohomology.

2.2.2. Rank one and pure torsion cases. Now suppose that $\delta : \mathbf{Q}_p^\times \to L^\times$ is a continuous character. We introduce some notation to make statements slightly shorter. We define $H^\bullet(\delta) := H^\bullet(\mathcal{R}_L(\delta))$ and $\widehat{T} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^\times, \mathbf{G}_{m/\mathbf{Q}_p})$. Inside $\widehat{T}$ we have two subsets

$$\widehat{S}^+ = \left\{ z^{-j} \colon 0 \leq j \right\},$$
$$\widehat{S}^- = \left\{ |z|\, z^i \colon i \geq 1 \right\}.$$

You can remember the decoration $\pm$ because the elements of $\widehat{S}^+$ have non-negative weight whereas the elements of $\widehat{S}^-$ have negative weight. Finally, we denote by $\widehat{T}_g = \widehat{T} \backslash (\widehat{S}^+ \cup \widehat{S}^-)$. These are the "generic characters" and the terminology is justified by the following calculation.

PROPOSITION 2.13. *Suppose that $\delta : \mathbf{Q}_p^\times \to L^\times$ is a character. Then,*

$$\dim_L H^0(\delta)) = \begin{cases} 0 & \delta \notin \widehat{S}^+ \\ 1 & \delta \in \widehat{S}^+ \end{cases},$$

$$\dim_L H^1(\delta) = \begin{cases} 1 & \delta \in \widehat{T}_g \\ 2 & \delta \notin \widehat{T}_g \end{cases}, \text{ and}$$

$$\dim_L H^2(\delta) = \begin{cases} 0 & \delta \notin \widehat{S}^- \\ 1 & \delta \in \widehat{S}^-. \end{cases}$$

PROOF. For $j = 0$ and $j = 1$ this is [**23**, Theorem 2.9] and for $j = 2$ this is [**46**, Proposition 2.12]. □

We are going to also need to make sure we know the cohomology of torsion (φ, Γ)-modules. For that, we need the following.

PROPOSITION 2.14. *If $\delta : \mathbf{Q}_p^\times \to L^\times$ is a continuous character then $H^2(\mathcal{R}_L(\delta)/t^r) = (0)$. If $j = 0$ or $j = 1$ then*

$$\dim_L H^j(\mathcal{R}_L(\delta)/t^r) = \begin{cases} 1 & \text{if } 0 \leq w(\delta) < r, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The case of $j = 0$ is [**23**, Proposition 2.18]—note, however, that we use different conventions then *loc. cit.* The case of $j = 1, 2$ follows from [**46**] (see Proposition 2.16(b)). □

COROLLARY 2.15. *Suppose that $\eta, \delta : \mathbf{Q}_p^\times \to L^\times$ are two continuous characters of integer weights $\mathrm{wt}(\eta) \leq \mathrm{wt}(\delta)$. Then, for each integer $r > \mathrm{wt}(\delta) - \mathrm{wt}(\eta)$, there is an exact sequence*

$$0 \to t^{r-(\mathrm{wt}(\delta)-\mathrm{wt}(\eta))}\mathcal{R}_L(\eta) \to \mathcal{R}_L(\eta) \to \mathcal{R}_L(\delta)/t^r \to \mathcal{R}_L(\delta)/t^{\mathrm{wt}(\delta)-\mathrm{wt}(\eta)} \to 0. \tag{2.3}$$

PROOF. By Proposition 2.14 in degree zero we have that if

$$\mathrm{wt}(\eta) - \mathrm{wt}(\delta) \leq m < r + (\mathrm{wt}(\eta) - \mathrm{wt}(\delta)) \tag{2.4}$$

then

$$\dim_L \mathrm{Hom}_{(\varphi,\Gamma)}(t^m \mathcal{R}_L(\eta), \mathcal{R}_L(\delta)/t^r) = 1.$$

If m is outside the range (2.4) then there are no (φ, Γ)-equivariant maps $t^m\mathcal{R}_L(\eta) \to \mathcal{R}_L(\delta)/t^r$. Thus, if $m \leq m'$ are two different integers then the natural restriction map

$$\mathrm{Hom}_{(\varphi,\Gamma)}(t^m \mathcal{R}_L(\eta), \mathcal{R}_L(\delta)/t^r) \to \mathrm{Hom}_{(\varphi,\Gamma)}(t^{m'} \mathcal{R}_L(\eta), \mathcal{R}_L(\delta)/t^r)$$

is either zero or an isomorphism for dimension reasons. In particular, up to a consistent choice of scalars in $L^\times$ we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t^{r-(\mathrm{wt}(\delta)-\mathrm{wt}(\eta))}\mathcal{R}_L(\eta) & \longrightarrow & \mathcal{R}_L(\eta) & \longrightarrow & \mathcal{R}_L(\delta)/t^r & & \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & t^{r-(\mathrm{wt}(\delta)-\mathrm{wt}(\eta))}\mathcal{R}_L(\eta) & \longrightarrow & t^{-(\mathrm{wt}(\delta)-\mathrm{wt}(\eta))}\mathcal{R}_L(\eta) & \longrightarrow & \mathcal{R}_L(\delta)/t^r & \longrightarrow & 0 \end{array}$$

whose rows are exact and with injective middle vertical arrow. This gives exactness of the first three terms in (2.3). For the final term, note that the injectivity of the middle vertical arrow implies that we have an isomorphism

$$\mathrm{coker}\,(\mathcal{R}_L(\eta) \to \mathcal{R}_L(\delta)/t^r) \cong \mathrm{coker}\left(\mathcal{R}_L(\eta) \to t^{-(\mathrm{wt}(\delta)-\mathrm{wt}(\eta))}\mathcal{R}_L(\eta)\right),$$

of $\mathcal{R}_L$-modules. The exactness of the final term of (2.3) easily follows now. □

2.2.3. Duality and the Euler characteristic formula. In the case of an L-linear representation V of $G_{\mathbf{Q}_p}$, one knows that $H^2(G_{\mathbf{Q}_p}, \chi_{\mathrm{cycl}})$ is one-dimensional and the cup product pairing

$$H^i(G_{\mathbf{Q}_p}, V) \times H^{2-i}(G_{\mathbf{Q}_p}, V^\vee(\chi_{\mathrm{cycl}})) \to H^2(G_{\mathbf{Q}_p}, \chi_{\mathrm{cycl}}) \cong \mathbf{Q}_p$$

is a non-degenerate perfect pairing (local Tate duality). We further have the Euler-Poincaré-Tate characteristic formula:

$$\chi(V) := \sum_{i=0}^{2} (-1)^i \dim_L H^i(G_{\mathbf{Q}_p}, V) = -\dim_L V.$$

These formulas are extemely important because reduces a questions about first degree cohomology (the most interesting one!) into questions about invariants of a fixed representation and a twist of its dual. Their analogies in the theory of (φ, Γ)-modules will be fundamental for our comptuations. Recall that a torsion (φ, Γ)-module is a generalized (φ, Γ)-module S such that $S[1/t] = (0)$.

PROPOSITION 2.16. [**46**, Theorems 4.3, 4.7] *Let S be a torsion (φ, Γ)-module and D a (φ, Γ)-module over $\mathcal{R}_L$. Then,*

(a) $H^2(\mathcal{R}_L(z\,|z|)) \cong \mathbf{Q}_p$.
(b) $H^2(S) = (0)$ *and* $\dim_L H^0(S) = \dim_L H^1(S)$.
(c) For each i, the pairing

$$H^i(D) \times H^{2-i}(D^\vee(z\,|z|)) \to \mathbf{Q}_p$$

given by (a) and the cup product is a non-degenerate perfect pairing.

(d) We have

$$-\operatorname{rank}_{\mathcal{R}_L} D = \sum_{i=0}^{2} (-1)^i \dim_L H^i(D).$$

Note that part (d) is true for S as well if we agree that a torsion (φ, Γ)-module has rank zero.

2.2.4. Bloch-Kato Selmer groups. Suppose that V is an L-linear representation of $G_{\mathbf{Q}_p}$. Then Bloch and Kato [**10**] have defined and studied subspaces $H^1_f(G_{\mathbf{Q}_p}, V)$ and $H^1_g(G_{\mathbf{Q}_p}, V)$ of the Galois cohomology $H^1(G_{\mathbf{Q}_p}, V)$ which, at least in the case that V is crystalline, parameterize extensions W of L by V which are crystalline. They play a local at p role in the construction of global Galois cohomology groups whose dimension is one of the terms in the so-called Bloch-Kato conjecture. In our deformation theory computations, we will need to consider these Selmer groups.

Of particular important to use will be the case where $D = \operatorname{ad} D_0$ is the adjoint module attached to a (φ, Γ)-module D_0.

Recall that in §2.1.2 we gave explicit descriptions of Fontaine's functors in terms of (φ, Γ)-modules. We have, as well, remarked that the cohomology group $H^1(D)$ of a (φ, Γ)-module is canonically isomorphic to the group of extensions $\operatorname{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_L, D)$. If c is a cocycle in $H^1(D)$ then we denote by D_c the extension

$$0 \to D \to D_c \to \mathcal{R}_L \to 0$$

of (φ, Γ)-modules corresponding to the cocycle c. Since $D_{\mathrm{dif}}(-)$ is an exact functor. Since taking Γ-invariants we get an exact sequence

$$0 \to D_{\mathrm{dR}}(D) \to D_{\mathrm{dR}}(D_c) \to L \xrightarrow{\partial^c_{\mathrm{dR}}} H^1(\Gamma, D_{\mathrm{dif}}(D)).$$

On the other hand, the functor $D_{\mathrm{dif}}(-)$ defines for us a canonical morphism of L-vector spaces

$$H^1(D) \xrightarrow{d_{\mathrm{dR}}} H^1(\Gamma, D_{\mathrm{dif}}(D)).$$

Similarly, the functor $D \mapsto D[1/t]$ is exact and so taking Γ-invariants, we get an exact sequence

$$0 \to D_{\mathrm{cris}}(D) \to D_{\mathrm{cris}}(D_c) \to L \xrightarrow{\partial^c_{\mathrm{cris}}} H^1(\Gamma, D[1/t]).$$

Again, the other part of the picture is a functorial (in D) map

$$H^1(D) \xrightarrow{d_{\mathrm{cris}}} H^1(\Gamma, D[1/t]).$$

We then define the Bloch-Kato Selmer groups in the context of (φ, Γ)-modules.

Definition. *The Bloch-Kato Selmer groups are*

$$H^1_f(D) := \ker\left(H^1(D) \xrightarrow{d_{\mathrm{cris}}} H^1(\Gamma, D[1/t])\right)$$
$$H^1_g(D) := \ker\left(H^1(D) \xrightarrow{d_{\mathrm{dR}}} H^1(\Gamma, D_{\mathrm{dif}}(D))\right)$$

Proposition 2.17. *Let $c \in H^1(D)$ be represented by an extension $0 \to D \to D_c \to \mathcal{R}_L \to 0$. Then $c \in H^1_f(D)$ (resp. $c \in H^1_g(D)$) if and only if the sequence*

$$0 \to D_?(D) \to D_?(D_c) \to L \to 0 \tag{2.5}$$

is still exact where $? = \mathrm{cris}$ for H^1_f and $? = \mathrm{dR}$ for H^1_g.

Proof. The proof is the same for either so let us just assume that we are working in the case of D_{cris}. It is easy to check that the relation between d_{cris} and ∂_{cris} is given by $d_{\mathrm{cris}}(c) = \partial^c_{\mathrm{cris}}(1)$. Thus, the sequence (2.5) is exact if and only if $\partial^c_{\mathrm{cris}}(1) = 0$, if and only if $c \in \ker(d_{\mathrm{cris}})$. ☐

The alternative description easily shows us that we have the following corollary.

Corollary 2.18. *Suppose that D is de Rham (resp. crystalline). Then $c \in H^1_g(D)$ (resp. $c \in H^1_f(D)$) if and only if D_c is de Rham (resp. crystalline).*

As one expects, this is all completely compatible with the usual theory coming from $G_{\mathbf{Q}_p}$-representations. We have as well analogs of the computations originally done by Bloch and Kato in that case (see [**10**, Corollary 3.8.4]).

Proposition 2.19 ([**6**, Proposition 1.4.2, Corollary 1.4.5]). *Suppose that V is an L-linear representation of $G_{\mathbf{Q}_p}$. The natural isomorphism $H^1(G_{\mathbf{Q}_p}, V) \cong H^1(D_{\mathrm{rig}}(V))$ induces isomorphisms*

$$H^1_f(G_{\mathbf{Q}_p}, V) \xrightarrow{\cong} H^1_f(D_{\mathrm{rig}}(V)), \text{ and}$$
$$H^1_g(G_{\mathbf{Q}_p}, V) \xrightarrow{\cong} H^1_g(D_{\mathrm{rig}}(V)).$$

Furthermore, if D is de Rham. Then,

$$\dim_L H^1_f(D) = \dim H^0(D) + \dim D_{\mathrm{dR}}(D)/D^+_{\mathrm{dR}}(D) \tag{2.6}$$

$$\dim_L H^1_g(D) = \dim H^1_f(D) + \dim D_{\mathrm{cris}}(D^\vee)^{\varphi=p}. \tag{2.7}$$

In particular, if D is crystalline and p^{-1} is not a φ-eigenvalue of $D_{\mathrm{cris}}(D)$ then $H^1_f(D) = H^1_g(D)$.

The reader should note that the the formulas are very easy to compute. For example, suppose that D is crystalline and that p^{-1} is not an eigenvalue for φ on $D_{\mathrm{cris}}(D)$. Then, $\dim_L H^1_f(D)$ is exactly the number of negative Hodge-Tate weights of D plus the number of (φ,Γ)-equivariant copies of $\mathcal{R}_L$ which are embedded in D.

2.3. Triangulations and parabolizations

We now move on to the special class of (φ,Γ)-modules we will study for the rest of this thesis: the trianguline (φ,Γ)-modules. The key feature we will exploit is that the Galois representations which arise from algebraic geometry become reducible (in many ways) when we pass to the world of (φ,Γ)-modules. Thus, we still denote by L a p-adic field and by A a finite L-algebra.

2.3.1. Parabolizations of (φ,Γ)-modules. Let D be a (φ,Γ)-module over $\mathcal{R}_A$ and we denote its rank by n.

Definition. *Fix an integer $1 \leq s \leq n$. A parabolization, of length s, of D is a strictly increasing filtration P*

$$P : 0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_s = D$$

such that

(a) each P_i is a (φ,Γ)-submodule of D over $\mathcal{R}_A$, and
(b) $\operatorname{coker}(P_{i-1} \to P_i)$ is a direct summand of P_i as a $\mathcal{R}_A$-module for each $i = 1,\dots,s$.

In the case that $s = n$ (i.e. we have a full filtration inside D) we call P a triangulation. In that case, Proposition 2.1 implies that each (φ,Γ)-module $\mathrm{Gr}_i\, P := P_i/P_{i-1}$ is rank one and the n-tuple $(\delta_1,\dots,\delta_n)$ of characters $\delta_i : \mathbf{Q}_p^\times \to A^\times$ such that

$$\mathrm{Gr}_i\, P \cong \mathcal{R}_A(\delta_i)$$

is called the parameter of the triangulation P.

Lemma 2.20. *Suppose D is a triangulated (φ,Γ)-module of rank n with parameter $(\delta_1,\dots,\delta_n)$. The list of Hodge-Tate-Sen weights of D are $\{\mathrm{wt}(\delta_1),\dots,\mathrm{wt}(\delta_n)\}$.*

Proof. The Hodge-Tate-Sen weights of D only depend on the semi-simplification of D (since that is true of the eigenvalues of Θ_{Sen} acting on $D_{\mathrm{Sen}}(D)$). □

Definition. *A (φ,Γ)-module is said to be trianguline if it has a triangulation.*

Using Galois representations arising from algebraic geometry, we can construct many examples of trianguline (φ,Γ)-modules and for each one, many possible triangulations. These different triangulations corresponding on the Galois side to refinements, which we now define. Assume for the moment that D is a crystalline (φ,Γ)-module over $\mathcal{R}_L$ where the eigenvalues of φ acting on $D_{\mathrm{cris}}(D)$ all live in $L^\times$.

Definition. *A partial refinement of D is the choice of a φ-stable filtration R of $D_{\mathrm{cris}}(D)$*

$$R : 0 = R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_s = D_{\mathrm{cris}}(D).$$

If $s = n$ (i.e. we have a full filtration) then we remove the word partial and just refer to R as a refinement.

If R is a refinement then it defines an ordering $(\phi_1, \ldots, \phi_n)$ of the crystalline eigenvalues by insisting that φ acting on R_i has eigenvalues $\{\phi_1, \ldots, \phi_i\}$. Further, if φ has distinct eigenvalues on $D_{\mathrm{cris}}(D)$ (which will be assumed in our applications) then this ordering is equivalent to a refinement. Moreover, we also obtain an ordering $(s_1, \ldots, s_n)$ of the Hodge-Tate weights (each of which is an integer since D is crystalline) by insisting the R_i has weights $\{s_1, \ldots, s_i\}$ with respect to the Hodge filtration $\mathrm{Fil}^\bullet D_{\mathrm{cris}}(D)$.

Thus, to a crystalline (φ, Γ)-module D we can consider either its triangulations or its refinements. In either case, there is invariant data attached to either:

$$\text{a triangulation } P \rightsquigarrow \text{the parameter } (\delta_1, \ldots, \delta_n)$$
$$\text{a refinement } R \rightsquigarrow \begin{array}{c}\text{the orderings of eigenvalues the } (\phi_1, \ldots, \phi_n)\\ \text{and the Hodge-Tate weights } (s_1, \ldots, s_n)\end{array}.$$

The relationship between parabolizations and partial refinements, along with the dictionary of passing between these invariants, is given by the following. Recall that we defined the unr notation in Example 2.3.

Proposition 2.21. *(a) Let D be a crystalline (φ, Γ)-module over $\mathcal{R}_L$ whose crystalline eigenvalues all live in $L^\times$. Then $P \mapsto D_{\mathrm{cris}}(P)$ induces bijections*

$$\{\textit{parabolizations of } D\} \leftrightarrow \{\textit{partial refinements of } D_{\mathrm{cris}}(D)\}, \textit{ and}$$
$$\{\textit{triangulations of } D\} \leftrightarrow \{\textit{refinements of } D_{\mathrm{cris}}(D)\}.$$

(b) If P is a triangulation with parameter $(\delta_1, \ldots, \delta_n)$ then the orderings associated to $D_{\mathrm{cris}}(P)$ are given by

$$(\phi_1, \ldots, \phi_n) = (p^{\mathrm{wt}(\delta_1)}\delta_1(p), \ldots, p^{\mathrm{wt}(\delta_n)}\delta_n(p)), \textit{ and}$$
$$(s_1, \ldots, s_n) = (\mathrm{wt}(\delta_1), \ldots, \mathrm{wt}(\delta_n)).$$

(c) If R is a refinement with orderings $(\phi_1, \ldots, \phi_n)$ and $(s_1, \ldots, s_n)$ as above, then the parameter $(\delta_1, \ldots, \delta_n)$ of the corresponding triangulation P is given by

$$(\delta_1, \ldots, \delta_n) = (z^{-s_1}\,\mathrm{unr}(\phi_1), \ldots, z^{-s_n}\,\mathrm{unr}(\phi_n)).$$

Proof. For the bijection in this generality, see [**16**, Lemma 3.10]. The correspondence between the parameters and the orderings of the eigenvalues/weights is explained in [**5**, Proposition 2.4.1] (where the triangulation case of part (a) is also proved). Note there that $z\big|_\Gamma = (z\,|z|)\big|_\Gamma$ and thus the formulas of *loc. cit.* are the same as ours. □

In order to illustrate the role that the weights are playing, and because this type of computation is fundamental to understanding the variation in p-adic families, we include the following worked example in dimension two.

Example 2.22. Consider what is happening for a two-dimensional crystalline (φ, Γ)-module. Assume that D is a crystalline with regular weights $k_1 < k_2$ and distinct eigenvalues $\{\phi, \phi'\}$. Without loss of generality, $D_{\mathrm{cris}}(D)^{\varphi=\phi} \neq \mathrm{Fil}^{k_2} D_{\mathrm{cris}}(D)$. Thus, the refinement with $\phi_1 = \phi$ orders the weights (k_1, k_2) and we get a triangulation[1]

$$0 \to \mathcal{R}_L(z^{-k_1}\,\mathrm{unr}(\phi)) \xrightarrow{i} D \xrightarrow{\pi} \mathcal{R}_L(z^{-k_2}\,\mathrm{unr}(\phi')) \to 0. \tag{2.8}$$

The other refinement corresponds to the ordering (ϕ', ϕ), but we need to have two cases in order to see what the ordering of the weights is.

[1] In the two-dimensional case, we often write a triangulation as a short exact sequence, the submodule being the first/only non-trivial step in the triangulation.

We claim that $\mathrm{wt}(\phi') := \mathrm{wt}(D_{\mathrm{cris}}(D)^{\varphi=\phi'}) = k_2$ if and only if the sequence is split (2.8) (as (φ,Γ)-modules; it is split as an $\mathcal{R}_L$-module already). One direction is clear, so assume that $\mathrm{wt}(\phi') = k_2$. Then, we look at the induced triangulation

$$0 \to \mathcal{R}_L(z^{-k_2}\,\mathrm{unr}(\phi')) \xrightarrow{j} D \to \mathcal{R}_L(z^{-k_1}\,\mathrm{unr}(\phi)) \to 0.$$

We claim that j is a section of π. Since

$$\dim_L \mathrm{Hom}_{(\varphi,\Gamma)}\Big(\mathcal{R}_L(z^{-k_2}\,\mathrm{unr}(\phi')), \mathcal{R}_L(z^{-k_2}\,\mathrm{unr}(\phi'))\Big) = 1$$

it suffices by way of contradiction to assume that that $\pi \circ j = (0)$. If that is the case then j defines a non-zero map $\mathcal{R}_L(z^{-k_2}\,\mathrm{unr}(\phi')) \to \mathcal{R}_L(z^{-k_1}\,\mathrm{unr}(\phi))$, which must be an isomorphism because $\mathrm{im}(j)$ is saturated inside D. However, $z^{-k_2}\,\mathrm{unr}(\phi')$ and $z^{-k_1}\,\mathrm{unr}(\phi)$ have different weights, so this is not possible. Thus, up to a factor of $L^\times$, we must have that j is a section of π and thus D is split as a (φ,Γ)-module.

Anyways, what it means is that the triangulation corresponding to (ϕ',ϕ) is one of two possibilities:

$$0 \to \mathcal{R}_L(z^{-k_2}(\mathrm{unr}(\phi')) \to D \to \mathcal{R}_L(z^{-k_1}(\mathrm{unr}(\phi))) \to 0 \qquad \text{(if } D \text{ is split)}$$

or

$$0 \to \mathcal{R}_L(z^{-k_1}(\mathrm{unr}(\phi')) \to D \to \mathcal{R}_L(z^{-k_2}(\mathrm{unr}(\phi))) \to 0 \qquad \text{(if } D \text{ is non-split)}.$$

If we call (δ_1,δ_2) the parameter corresponding to (ϕ,ϕ') then the parameter of (ϕ',ϕ) is either (δ_2,δ_1) if D is split, or

$$(z^{k_2-k_1}\delta_2, z^{k_1-k_2}\delta_1) = (z^{\mathrm{wt}(\delta_2)-\mathrm{wt}(\delta_1)}\delta_2, z^{\mathrm{wt}(\delta_1)-\mathrm{wt}(\delta_2)}\delta_1)$$

if D is non-split.

The above example proves the following, which we record for later use:

Lemma 2.23. *Suppose that D is a rank two crystalline (φ,Γ)-module of regular weight and with distinct crystalline eigenvalues contained in $L^\times$. If (δ,η) is the parameter of some triangulation of D then the other is given by*

$$\begin{cases} (\eta,\delta) & \text{if } D \text{ is split,} \\ (z^{\mathrm{wt}(\eta)-\mathrm{wt}(\delta)}\eta, z^{\mathrm{wt}(\delta)-\mathrm{wt}(\eta)}\delta) & \text{if } D \text{ is non-split.} \end{cases}$$

Revisiting the example one more time, we see that the difference between the ordering of the weights corresponding to (ϕ',ϕ) was whether or not the line $D_{\mathrm{cris}}(D)^{\varphi=\phi'}$ had the highest weight. Focusing instead on the weight first, and the action of φ second, this is a question of whether or not the line $\mathrm{Fil}^{k_2} D_{\mathrm{cris}}(D)$ is φ-stable. This leads us to a definition:

Definition. *Assume that D is crystalline. Then, a saturated (φ,Γ)-submodule $P \subset D$ is called non-critical if there exists an integer k such that $D_{\mathrm{cris}}(P) \oplus \mathrm{Fil}^k D_{\mathrm{cris}}(D) = D_{\mathrm{cris}}(D)$. We say that P is critical if it is not non-critical.*

We remark that D itself is always a non-critical submodule of itself. In fact, if k is the highest Hodge-Tate weight of D then $D_{\mathrm{cris}}(D) = D_{\mathrm{cris}}(D) \oplus \mathrm{Fil}^{k+1} D_{\mathrm{cris}}(D)$. In general, whether a (φ,Γ)-submodule P of D is critical or not depends on the Hodge-Tate weights of P (relative to the Hodge-Tate weights of D).

Lemma 2.24. *Suppose that D is crystalline and that the Hodge-Tate weights $k_1 < k_2 < \cdots < k_n$ of D are distinct. Let $P \subset D$ be a saturated (φ,Γ)-submodule of rank m. The following are equivalent:*

(a) P is non-critical;

(b) $D_{\mathrm{cris}}(P) \oplus \mathrm{Fil}^{k_{m+1}} D_{\mathrm{cris}}(D) = D_{\mathrm{cris}}(D)$;
(c) *the Hodge-Tate weights of* P *are the lowest* m *weights;*
(d) $\det P \subset \wedge^m D$ *is non-critial.*

PROOF. It is clear that (b) and (c) are equivalent and that (b) implies (a). Conversely, assume that P is non-critical and choose k such that $D_{\mathrm{cris}}(P) \oplus \mathrm{Fil}^k D_{\mathrm{cris}}(D) = D_{\mathrm{cris}}(D)$. Since P is crystalline (Lemma 2.11) we get that $d - m = \dim_L \mathrm{Fil}^k D_{\mathrm{cris}}(D)$. Thus, by regularity of the Hodge-Tate weights, we get $\mathrm{Fil}^k D_{\mathrm{cris}}(D) = \mathrm{Fil}^{k_{m+1}} D_{\mathrm{cris}}(D)$ and so (a) implies (b).

We only have to prove that (c) and (d) are equivalent. Notice that (c) is equivalent to the single Hodge-Tate weight of $\det P$ being $k_1 + \cdots + k_m$. Moreover, even though $\wedge^m D$ need not have distinct Hodge-Tate weights, the lowest weight $k_1 + \cdots + k_m$ is multiplicity one, by the distinctness of the k_i. Thus $\det P$ has lowest weight $k_1 + \cdots + k_m$ if and only if

$$D_{\mathrm{cris}}(\det P) \oplus \mathrm{Fil}^{k_1+\cdots+k_{m-1}+k_{m+1}} D_{\mathrm{cris}}(\wedge^m D) = D_{\mathrm{cris}}(\wedge^m D).$$

This shows the conclusion. □

Notice that the lemma provides us with a way to check that an entire triangulation is non-critical.

EXAMPLE 2.25. Suppose that $D = D_{\mathrm{rig}}(V)$ with V a crystalline representation with regular Hodge-Tate weights $k_1 < k_2 < \cdots < k_n$. Suppose as well that we order the eigenvalues $(\phi_1, \ldots, \phi_n)$ of φ acting on $D_{\mathrm{cris}}(D)$ such that

$$v_p(\phi_1) < k_2$$
$$v_p(\phi_1 \cdots \phi_i) < k_1 + \cdots k_{i-1} + k_{i+1}.$$

Then, the refinement (or, rather, triangulation) defined by $(\phi_1, \ldots, \phi_n)$ is non-critical. Indeed, $D_{\mathrm{cris}}(D)$ is weakly admissible since it comes from a crystalline Galois representation. This implies that the φ-stable line $D_{\mathrm{cris}}(D)^{\varphi=\phi_1}$ defines a Hodge weight

$$s_1 \leq v_p(\phi_1) < k_2.$$

Since s_1 must be some weight k_i it follows that $s_1 = k_1$. Similarly, $s_i = k_i$ for each i. We refer to the ordering $(\phi_1, \ldots, \phi_n)$ and V as being *numerically non-critical* (see [**5**, Remark 2.4.6] as well).

Finally, we end this subsection with a construction that isolates out the pieces of a triangulation which we will expect to vary in p-adic families. We continue to let D be a crystalline (φ, Γ)-module of rank n.

DEFINITION. *Let* P *be a triangulation of* D. *We let*

$$I^{\mathrm{nc}} := \{i \colon P_i \text{ is non-critical}\} = \{i_1 < i_2 < \ldots < i_s\}.$$

We define the maximal non-critical parabolization P^{nc} *as the filtration*

$$P^{\mathrm{nc}} : 0 \subsetneq P_{i_1} \subsetneq P_{i_2} \subsetneq \cdots \subsetneq P_{i_s} = D.$$

The remarks preceding Lemma 2.24 show that we are justified in knowing that the top index i_s gives D. At least in the case of a (φ, Γ)-module D which has regular weights, we see that an alternative definition is that P_i is a step in the filtration P^{nc} if and only if the weights of P_i are the lowest possible ones. The definition of non-critical/critical for a single (φ, Γ)-submodule of D extends to triangulations.

DEFINITION. *Suppose that* P *is a triangulation of* D. *Then, we say that* P *is non-critical if* $P^{\mathrm{nc}} = P$. *Otherwise, we say that* P *is critical. At the opposite end, we say that* P *is fully critical if* P^{nc} *is the trivial filtration* $0 \subsetneq D$.

In Example 2.22 we saw that if D is crystalline of rank two then D is non-split if and only if both possible triangulations are non-critical. The use of the word maximal in the previous definition is justified by the following lemma.

LEMMA 2.26. *Suppose P is a triangulation of D and fix $1 \leq j \leq n$. Then, the induced triangulation on $\mathrm{Gr}_j P^{\mathrm{nc}}$ is fully critical.*

PROOF. Let P have non-critial jumps $\{i_1 < i_2 < \ldots < i_s\}$. The induced triangulation on $\mathrm{Gr}_j P^{\mathrm{nc}} = P_{i_j}/P_{i_{j-1}}$ is given by

$$0 \subsetneq P_{i_{j-1}+1}/P_{i_{j-1}} \subsetneq \cdots \subsetneq P_{i_j-1}/P_{i_{j-1}} \subsetneq P_{i_j}/P_{i_{j-1}} = \mathrm{Gr}_j P^{\mathrm{nc}}.$$

Since P_{i_j} and $P_{i_{j-1}}$ are non-critical, there are integers $k' \leq k$ such that

$$\begin{aligned} D_{\mathrm{cris}}(\mathrm{Gr}_j P^{\mathrm{nc}}) &= \Big(D_{\mathrm{cris}}(D)/\operatorname{Fil}^k D_{\mathrm{cris}}(D)\Big) \,/\, \Big(D_{\mathrm{cris}}(D)/\operatorname{Fil}^{k'} D_{\mathrm{cris}}(D)\Big) \\ &= \operatorname{Fil}^{k'} D_{\mathrm{cris}}(D)/\operatorname{Fil}^k D_{\mathrm{cris}}(D). \end{aligned}$$

Thus if $k' \leq m \leq k$ we have that

$$D_{\mathrm{cris}}(\mathrm{Gr}_j P^{\mathrm{nc}})/\operatorname{Fil}^m D_{\mathrm{cris}}(\mathrm{Gr}_j P^{\mathrm{nc}}) = \operatorname{Fil}^{k'} D_{\mathrm{cris}}(D)/\operatorname{Fil}^m D_{\mathrm{cris}}(D).$$

Now suppose that $i_{j-1} < r < i_j$. Then,

$$D_{\mathrm{cris}}(P_r/P_{i_{j-1}}) = \operatorname{Fil}^{k'} D_{\mathrm{cris}}(D)/\operatorname{Fil}^m D_{\mathrm{cris}}(D)$$

if and only if

$$\begin{aligned} D_{\mathrm{cris}}(P_r) &= \operatorname{Fil}^{k'} D_{\mathrm{cris}}(D)/\operatorname{Fil}^m D_{\mathrm{cris}}(D) \oplus D_{\mathrm{cris}}(D)/\operatorname{Fil}^{k'} D_{\mathrm{cris}}(D) \\ &= D_{\mathrm{cris}}(D)/\operatorname{Fil}^m D_{\mathrm{cris}}(D). \end{aligned}$$

Since P_r is assumed to be critical, this is impossible. □

REMARK. In the case of regular weight, one could use the description given in Lemma 2.24 to see the result above immediately.

2.3.2. Some cohomology computations of triangulated (φ, Γ)-modules. To end this chapter, we are going to collect some easy calculations of the cohomology of trianguline (φ, Γ)-modules. These will provide us with a reference to call on for our later study of families of (φ, Γ)-modules.

Recall that the computations of Colmez and Liu (Proposition 2.13) give us completely the cohomology of all the rank one (φ, Γ)-modules. Thus, in most cases, we can compute the cohomology of a trianguline (φ, Γ)-module. Recall that we defined the generic characters $\widehat{\mathcal{T}}_g$ on page 34. They are all the characters $\delta : \mathbf{Q}_p^\times \to L^\times$ which are not of the form z^{-j} for $j \geq 0$ or $|z|\, z^i$ for $i \geq 1$.

DEFINITION. *Let D be a trianguline (φ, Γ)-module. We say that D is almost generic if there is a triangulation P of D such that the associated parameter is of the form $(\delta_1, \ldots, \delta_n)$ where $\delta_1 \in x^{\mathbf{Z}}$ and $\delta_j \in \widehat{\mathcal{T}}_g$ for $j \geq 2$. Any parameter which satisfies this condition we call an almost generic parameter for D and we say that D is almost generic with respect to this parameter.*

EXAMPLE 2.27. If D is a rank two trianguline (φ, Γ)-module which sits as an extension

$$0 \to \mathcal{R}_L(\delta_1) \to D \to \mathcal{R}_L(\delta_2) \to 0,$$

and $\delta_2\delta_1^{-1} \in \widehat{\mathcal{T}}_g$ then $D(\delta_1^{-1})$ is almost generic with respect to the parameter $(\mathbf{1}, \delta_2\delta_1^{-1})$. In fact, this is how we will most often use the definition.

LEMMA 2.28. *Suppose that D is almost generic and choose an almost generic parameter $(\delta_1, \ldots, \delta_n)$ for D. Then,*

$$\dim_L H^j(D) = \begin{cases} \dim H^0(\mathcal{R}_L(\delta_1)) & \text{if } j = 0 \\ n + \dim H^0(\mathcal{R}_L(\delta_1)) & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

PROOF. By the almost generic assumption and Proposition 2.13 we have that

$$H^0(\mathcal{R}_L(\delta_j)) = H^2(\mathcal{R}_L(\delta_j)) = (0), \qquad j = 2, \ldots, n,$$

and

$$H^2(\mathcal{R}_L(\delta_1)) = (0).$$

In particular, if P is the associated triangulation then it follows by induction that

- $H^2(D) = (0)$,
- $H^0(\mathcal{R}_L(\delta_1)) \xrightarrow{\cong} H^0(D)$, and
- for each $i = 2, \ldots, n$ there is a short exact sequence

$$0 \to H^1(P_{i-1}) \to H^1(P_i) \to L \to 0.$$

The first two points cover the $j = 0$ and $j = 2$ computations. The case of $j = 1$ follows from the third point by induction on i and using that $\dim_L H^1(\mathcal{R}_L(\delta_1)) = 1 + \dim_L H^0(\mathcal{R}_L(\delta_1))$ since $\delta \in x^{\mathbf{Z}}$. □

We also have the following computation that will be used in Chapter 4 (see Lemma 4.15).

LEMMA 2.29. *Suppose that D is a crystalline (φ, Γ)-module of rank n with distinct crystalline eigenvalues. Choose a triangulation P for D and denote its parameter by $(\delta_1, \ldots, \delta_n)$. Assume that the lowest Hodge-Tate weight k of D is simple and choose i such that $\mathrm{wt}(\delta_i) = k$. Then, P_i is split $P_i \cong P_{i-1} \oplus \mathcal{R}_L(\delta_i)$.*

PROOF. By Lemma 2.11, each step P_i of P is crystalline. The (φ, Γ)-module P_i defines an extension class in $H^1_f(P_{i-1}(\delta_i^{-1}))$. We claim that this group is zero.

It follows from the distinctness of the crystalline eigenvalues and Proposition 2.13 that if $i \neq j$ then

$$\dim_L \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta_j)) = (0).$$

In particular, $\dim_L H^0(P_{i-1}(\delta_i^{-1})) = (0)$. By Proposition 2.19 we deduce that the dimension $\dim_L H^1_f(P_{i-1}(\delta_i^{-1}))$ is the number of negative Hodge-Tate weights of the (φ, Γ)-module $P_{i-1}(\delta_i^{-1})$. By the choice of i, there are none and thus $H^1_f(P_{i-1}(\delta_i^{-1})) = (0)$. □

CHAPTER 3

Deformation theory of (φ, Γ)-modules

We are now going to shift our focus towards the deformation theory of (φ, Γ)-modules. That is, we fix a (φ, Γ)-module $\mathcal{R}_L$ and study lifts of D to finite L-algebras (for specifics, see below). Our purpose is to develop and study deformation problems which will give, following Chapters 4 and 5, upper bounds on the dimensions of tangent spaces of p-adic families.

Our first goal will be to recall the setup of Mazur's deformation theory [**48**]. The theory under discussion was developed for Galois representations, but, we work instead with (φ, Γ)-modules. We do not claim to give any original ideas to the setup, but we hope that the reader will enjoy a rather self-contained exposition of the formalism. In particular, we carefully explain the definitions of tangent spaces and we will recall the deformation conditions arising from p-adic Hodge theory.

Our second goal in this chapter is to explain two deformation conditions which occur in p-adic families and to give estimates for the size of their deformation rings. The first such condition is the *paraboline deformation condition.* This is a direct generalization of the trianguline deformations studied in [**5**, Chapter 2]. However, for all the results of Chapter 5 which cannot be deduced from *loc. cit.* already, the paraboline deformations will not be enough. Thus, we study an extra deformation condition which we have coined *Kisin-type*. At non-critical points in a family, the Kisin-type deformations are the same as the trianguline deformations but at critically points, it imposes an extra condition. Putting the two conditions together, we are able to prove a theorem contingent on the computation of Kisin-type deformations rings in the fully critical case (recall the definition on page 40). We include two such computations in §3.3.

Throughout this chapter, we denote by $\mathfrak{AR}_L$ the category of local Artin L-algebras with residue field L. For a ring morphism $A \rightarrow A'$ to be a morphism in $\mathfrak{AR}_L$ it must induce the identity map on the residue fields. The typical element of $\mathfrak{AR}_L$ to keep in mind is $L[[x]]/(x^n)$ and when $n = 2$ we denote this ring, called the dual numbers, by $L[\varepsilon]$. Since we are going to use this notation in §3.1, let us denote[1] by $\mathfrak{C}_L$ the category of complete local noetherian L-algebras with residue field L. We restrict morphisms as before. If $A \in \mathfrak{C}_L$ with maximal ideal $\mathfrak{m}_A$ then $A/\mathfrak{m}_A^n$ is an element of $\mathfrak{AR}_L$ for all $n \geq 1$ and $A \cong \varprojlim A/\mathfrak{m}_A^n$.

3.1. Functors on $\mathfrak{AR}_L$

In order to not multiply statements in the sequel, we feel it is necessary to make precise what we will mean about representability. We also make an abstract definition of the Zariski tangent space. Fix a functor $\mathfrak{X} : \mathfrak{AR}_L \rightarrow \underline{\mathrm{Set}}$.

DEFINITION. *We say that $\mathfrak{X}$ is pro-representable if there exists an element $R \in \mathfrak{C}_L$ such that*

$$\mathrm{Hom}_{\mathfrak{C}_L}(R, A) = \mathfrak{X}(A)$$

for all A in $\mathfrak{AR}_L$.

[1]For the reader interested in the taxonomy, the letter 'C' stands for coefficient, as in coefficient ring. This was the original terminology used by Mazur. The use of 'AR' is something I picked up from Kisin's work.

We could extend the functor $\mathfrak{X}$ to a functor $\mathfrak{X}_0 : \mathfrak{C}_L \to \underline{\mathrm{Set}}$ by the formula

$$\mathfrak{X}_0(A) = \varprojlim \mathfrak{X}(A/\mathfrak{m}_A^n). \tag{3.1}$$

In that case, $\mathfrak{X}$ is pro-representable if and only if $\mathfrak{X}_0$ is representable in $\mathfrak{C}_L$, in the usual sense. Thus, we hereafter drop the 'pro' part of pro-representable and just refer to functors on $\mathfrak{AR}_L$ as being representable.

Remark. In many cases, one actually begins with a functor $\mathfrak{X}_0$ on $\mathfrak{C}_L$ satisfying (3.1) with $\mathfrak{X} := \mathfrak{X}_0\big|_{\mathfrak{AR}_L}$—in this case we call $\mathfrak{X}_0$ continuous. For example, all of the deformation functors coming from *Galois representations* behave this way. However, this is not always the case. We must restrict to only functors on $\mathfrak{AR}_L$ is because we have not considered (φ, Γ)-modules with coefficients over a general element of $\mathfrak{C}_L$.

Suppose that A, B and C are elements of $\mathfrak{AR}_L$ and $f : A \to C$, $g : B \to C$ are morphisms. Then, the fibered product is defined as

$$A \times_C B := \{(a, b) \in A \times B \colon f(a) = g(b)\}$$

and it is[2] naturally an element of $\mathfrak{AR}_L$. Clearly, a necessary condition for $\mathfrak{X}$ to be representable is that the natural map

$$\mathfrak{X}(A \times_C B) \to \mathfrak{X}(A) \times_{\mathfrak{X}(C)} \mathfrak{X}(B) \tag{3.2}$$

is a bijection. Moreover, Grothendieck's representability theorem [**33**] says that this is sufficient provided

- $\mathfrak{X}(L)$ is just a single element, and
- the Zariski tangent space $\mathfrak{X}(L[\varepsilon])$ is a finite-dimensional L-vector space (it is a vector space by a particular instance of (3.2)—see §3.2).

Regardless of how hard the condition (3.2) is to check, the questions we will be most interested in are not necessarily those of representability, but rather relative representability. To define this, suppose as well that $\mathfrak{X}' \subset \mathfrak{X}$ is a subfunctor.

Definition. *We say that $\mathfrak{X}'$ is relatively representable if for all choices of A, B and C, the diagram*

$$\begin{array}{ccc} \mathfrak{X}'(A \times_C B) & \longrightarrow & \mathfrak{X}'(A) \times_{\mathfrak{X}'(C)} \mathfrak{X}'(B) \\ \downarrow & & \downarrow \\ \mathfrak{X}(A \times_C B) & \longrightarrow & \mathfrak{X}(A) \times_{\mathfrak{X}(C)} \mathfrak{X}(B) \end{array} \tag{3.3}$$

is cartesian.

We have the following other criterion to be relatively representable. Note that this forces one to check (3.2) for now a huge set of functors. In the case of deformations of (φ, Γ)-modules, we will have a much short criterion (see Proposition 3.4)

Proposition 3.1. *Let $\mathfrak{X}' \subset \mathfrak{X}$ be a subfunctor. Then, $\mathfrak{X}'$ is relatively representable if and only if for every representable functor $\mathfrak{Y} : \mathfrak{AR}_L \to \underline{\mathrm{Set}}$ and morphism $\mathfrak{Y} \to \mathfrak{X}$, the functor $\mathfrak{Y}' := \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{Y}$ is representable.*

Proof. This follows easily from the definitions of fiber products and Grothendieck's theorem. ∎

[2]This statement is false for $\mathfrak{C}_L$, see [**49**, p. 270].

3.2. A bestiary of deformation problems

We now specialize to the case of (φ, Γ)-modules. Fix $L/\mathbf{Q}_p$ a p-adic field. Unless specified, all (φ, Γ)-modules are over $\mathcal{R}_L$. We will use D to denote a (φ, Γ)-module.

DEFINITION. *Let D be a (φ, Γ)-module over $\mathcal{R}_L$. A deformation D_A to A is a (φ, Γ)-module over $\mathcal{R}_A$ and an isomorphism $\pi : D_A \otimes_A L \cong D$ of (φ, Γ)-modules.*

If (D_A, π) and (D'_A, π') are two deformations of D then a morphism $D_A \xrightarrow{f} D'_A$ is a morphism of (φ, Γ)-modules over $\mathcal{R}_A$ making the diagram

$$\begin{array}{ccc} D_A & \xrightarrow{\quad f \quad} & D'_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & D & \end{array}$$

commute. Two deformations are equivalent if there is an isomorphism between them.

If $A \in \mathfrak{AR}_L$ we denote by

$$\mathfrak{X}_D(A) = \{\text{deformations } D_A \text{ of } D \text{ to } A\} / \cong .$$

If $A \to A'$ is a map in $\mathfrak{AR}_L$ and D_A is a deformation of D to A then $D_{A'} := D_A \otimes_A A'$ is a deformation of D to A'. Thus $\mathfrak{X}_D$ defines a functor $\mathfrak{X}_D : \mathfrak{AR}_L \to \underline{\text{Set}}$ called the deformation functor of D. If V is an L-linear representation of $G_{\mathbf{Q}_p}$ then we also have the formal deformation functor $\mathfrak{X}_V : \mathfrak{AR}_L \to \underline{\text{Set}}$ defined by Mazur. Recall that by Proposition 1.12, for any element A of $\mathfrak{AR}_L$ the category of A-linear representations of $G_{\mathbf{Q}_p}$ is equivalent to a subcategory of (φ, Γ)-modules over A.

LEMMA 3.2. *The functor D_{rig} induces a natural transformation*

$$\mathfrak{X}_V \xrightarrow{D_{\text{rig}}} \mathfrak{X}_{D_{\text{rig}}(V)}$$

which is an isomorphism of functors $\mathfrak{AR}_L \to \underline{\text{Set}}$.

PROOF. The only point we have to make is that if D_A is a deformation of $D = D_{\text{rig}}(V)$ to A then D_A arises from a Galois representation. However, if we choose a composition series for A as a module over itself we see immediately that D_A is a successive extension of D by itself. Then, by Proposition 1.12 we have that D_A corresponds to a Galois representation. □

We explain now the Zariski tangent space associated to the functor $\mathfrak{X}_D$ and then give some examples of relatively representable subfunctors coming from p-adic Hodge theory. However, in order to make sense of the tangent space without a representability hypothesis we need the "tangent space hypothesis" $(\mathbf{T}_L)$ in notation of Mazur [**49**, §18] (see also the proof of Proposition 23.3(a) in *loc. cit.*).

LEMMA 3.3. *Suppose that D is a (φ, Γ)-module over $\mathcal{R}_L$. Then (3.2) holds for $\mathfrak{X} = \mathfrak{X}_D$, $A = B = L[\varepsilon]$ and $C = L$.*

PROOF. THis is deduced in the course of [**16**]. As it is only a small part of that proof, we include an argument for the reader here. Let $A = L[\varepsilon] \times_L L[\varepsilon]$. Let us denote the map (3.2) by

$$\mu = (\mu_1, \mu_2) : \mathfrak{X}_D(A) \to \mathfrak{X}_D(L[\varepsilon]) \times \mathfrak{X}_D(L[\varepsilon]).$$

Suppose that (D_1, D_2) is in the target of μ. We need to define the structure of (φ, Γ)-module on $D \otimes_L A$ which projects onto the structures for D_i. To this, consider a basis e_j for D as a (φ, Γ)-module—we get a basis $e_j \otimes 1$ for $D \otimes_L A$ and D_i. If $x \in \{\varphi, \gamma \in \Gamma\}$ denote $M_x^{(i)} \in M_n(\mathcal{R}_{L[\varepsilon]})$ the matrix for x acting on D_i in the basis $\{e_j \otimes 1\}_j$ for $i = 1, 2$. If we know that $M_x^{(1)} \equiv M_x^{(2)} \bmod \varepsilon$ for each x then the pairs $(M_x^{(1)}, M_x^{(2)})$ define elements of $M_n(\mathcal{R}_A)$ which give rise to a sufficient (φ, Γ)-module structure on $D \otimes_L A$. We claim that replacing D_1 by an isomorphic deformation, we can always achieve this. Indeed, since we have identifications $D_1/\varepsilon D_1 \cong D \cong D_2/\varepsilon D_2$ (by definition of fibered product) we have an element $f \in \mathrm{GL}_n(\mathcal{R}_L)$ which appropriately intertwines $M_x^{(1)}$ with $M_x^{(2)}$ modulo ε. But, $L[\varepsilon] \to L$ admits a section and so we can lift the element f to replace $M_x^{(1)}$ by $x(f)M_x^{(1)}f^{-1}$ to define an isomorphic deformation D_1 and such that D_1 and D_2 have the same (φ, Γ)-module structures modulo ε, as we claimed.

Now suppose that $\widetilde{D}$ and $\widetilde{D}'$ are elements of $\mathfrak{X}_D(A)$ such that $\mu(\widetilde{D}) = \mu(\widetilde{D}')$. Then for $i = 1, 2$ there exists an isomorphism $f_i : \mu_i(\widetilde{D}) \to \mu_i(\widetilde{D}')$ of (φ, Γ)-modules over $\mathcal{R}_{L[\varepsilon]}$ such that $f_i \equiv \mathrm{id} \bmod \varepsilon$. After choosing a basis for $\widetilde{D}$ and $\widetilde{D}'$ over $\mathcal{R}_A$, we can see f_i as an element of $\mathrm{GL}_n(\mathcal{R}_{L[\varepsilon]})$. Since we obviously have

$$\mathrm{GL}_n(\mathcal{R}_A) = \mathrm{GL}_n(\mathcal{R}_{L[\varepsilon]}) \times_{\mathrm{GL}_n(\mathcal{R}_L)} \mathrm{GL}_n(\mathcal{R}_{L[\varepsilon]})$$

we can see the element (f_1, f_2) as an element $f \in \mathrm{GL}_n(\mathcal{R}_A)$. Indeed, they each are the identity in $\mathrm{GL}_n(\mathcal{R}_L)$. Then (remember we chose this basis) the element f provides us with an isomorphism for $\widetilde{D}$ with $\widetilde{D}'$. □

Following this result there is now a canonical structure of L-vector space on the set $\mathfrak{X}_D(L[\varepsilon])$. First, if $\alpha \in L$ then the map

$$a + b\varepsilon \mapsto a + \alpha b\varepsilon : L[\varepsilon] \to L[\varepsilon]$$

morphism in $\mathfrak{AR}_L$ and thus defines by functoriality a scalar multiplication

$$\alpha : \mathfrak{X}_D(L[\varepsilon]) \to \mathfrak{X}_D(L[\varepsilon]).$$

This didn't require Lemma 3.3. We have as well the morphism

$$(a + b\varepsilon, a + c\varepsilon) \mapsto a + (b + c)\varepsilon : L[\varepsilon] \times_L L[\varepsilon] \xrightarrow{+} L[\varepsilon]$$

inside $\mathfrak{AR}_L$. Thus by functoriality and Lemma 3.3 we have a morphism

$$\mathfrak{X}_D(L[\varepsilon]) \times \mathfrak{X}_D(L[\varepsilon]) \overset{(3.2)}{\cong} \mathfrak{X}_D(L[\varepsilon] \times_L L[\varepsilon]) \xrightarrow{+} \mathfrak{X}_D(L[\varepsilon])$$

which defines addition. The two operations put the structure of vector space on $\mathfrak{X}_D(L[\varepsilon])$. Notice that the zero object in this vector space is the constant deformation[3] $D \oplus D$ where ε acts trivially.

Definition. *The Zariski tangent space to the deformation space $\mathfrak{X}_D$ of D is $\mathfrak{t}_D := \mathfrak{X}_D(L[\varepsilon])$.*

Remark. Notice that if $\mathfrak{X}' \subset \mathfrak{X}_D$ is a subfunctor which is relatively representable then the natural map

$$\mathfrak{X}'(L[\varepsilon] \times_L L[\varepsilon]) \to \mathfrak{X}'(L[\varepsilon]) \times \mathfrak{X}'(L[\varepsilon])$$

is still a bijection (this follows from $\mathfrak{X}'$ satisfying (3.3)). In particular, $\mathfrak{t}_{\mathfrak{X}'} := \mathfrak{X}'(L[\varepsilon])$ is closed under addition and thus defines a subspace $\mathfrak{t}_D$. We refer to it as the Zariski tangent space of the subfunctor $\mathfrak{X}'$.

As with the case of Galois representations we have the realization of $\mathfrak{t}_D$ as a certain Galois cohomology group. We denote by $\mathrm{ad}\, D$ the adjoint (φ, Γ)-module

$$\mathrm{ad}\, D = \mathrm{End}_{\mathcal{R}_L}(D) = D \otimes_{\mathcal{R}_L} D^\vee.$$

[3] Be warned, there is some ambiguity in the use of the word "split" because of this. On the one hand, we might see $\widetilde{D}$ inside $H^1(\mathrm{ad}\, D)$ (see below) and call it split if it is zero, i.e. the trivia deformation $D \oplus D$. On the other hand, it might also be "split" in the sense that it is a sum of characters. We hope that the context makes this clear.

If $\widetilde{D}$ is an element of $\mathfrak{t}_D$ then since $L[\varepsilon]$ has a composition series $0 \to L \to L[\varepsilon] \to L \to 0$ we see that $\widetilde{D}$ sits inside a short exact sequence

$$0 \to D \to \widetilde{D} \to D \to 0$$

of (φ, Γ)-modules over $\mathcal{R}_L$; the submodule is identified with $\varepsilon\widetilde{D}$ and the quotient with $\widetilde{D}/\varepsilon\widetilde{D}$. This defines isomorphisms

$$\mathfrak{t}_D \cong \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) \cong H^1(\mathrm{ad}\, D). \tag{3.4}$$

In the future, we will use both the cohomological description and the realization as extension classes to study $\mathfrak{t}_D$. For example, notice now that we we can reasonably expect to calculate $\mathfrak{t}_D$ using the Euler-Poincaré-Tate characteristic formula (Proposition 2.16).

So far we have not described when we can expect to get a representable object out of the theory of deforming (φ, Γ)-modules. For that we have the following criterion:

PROPOSITION 3.4. *To check that $\mathfrak{X}' \subset \mathfrak{X}$ is relatively representable it suffices to show:*

(i) if $f : A \to A'$ and an element $D_A \in \mathfrak{X}_D(A)$ is in $\mathfrak{X}'(A)$ then $D_{A'} := D_A \otimes_A A' \in \mathfrak{X}'(A')$ as well,
(ii) The case of (3.3) where $C = L$, and
(iii) If $f : A \to A'$ is injective and $D_A \in \mathfrak{X}_D(A)$ such that $D_A \otimes_A A' \in \mathfrak{X}'(A')$ then $D_A \in \mathfrak{X}'(A)$.

PROOF. The proof of this relies on Schlessinger's criterion and is explained in [**49**, §23] but see also the proof of [**5**, Proposition 2.3.9]. ☐

We now begin now giving examples of relatively representable deformation problems. In each case, we compute as well the corresponding Zariski tangent space.

3.2.1. Deformations arising from p-adic Hodge theory. Suppose first that D is Hodge-Tate and let k be a Hodge-Tate weight of D. For simplicity let us assume that k has multiplicity one. If D_A is a deformation of D to A then there is some Hodge-Tate-Sen weight $\kappa \in A$ such that $\kappa \equiv k \bmod \mathfrak{m}_A$. We say that D_A has a constant weight k if $\kappa = k$. We then define

$$\mathfrak{X}_D^k(A) = \left\{ D_A \in \mathfrak{X}_D(A) \colon D_A \text{ has constant Hodge-Tate-Sen weight } k \right\}.$$

Since $\mathbf{Z} \subset A$ for all A we clearly see that $\mathfrak{X}_D^k$ satisfies the conditions of Proposition 3.4. Thus $\mathfrak{X}_D^k$ is a relatively representable subfunctor of $\mathfrak{X}_D$.

To compute its Zariski tangent space, we revisit the definitions given in §2.1.2. Let $\widetilde{D} \in \mathfrak{X}_D^k(L[\varepsilon])$ be a deformation of D in the tangent space. We denote by $D_{\mathrm{Sen}}(\widetilde{D})_{(k)}$ the *generalized* eigenspace for the operator Θ_{Sen} with respect to the eigenvalue k (everything is seen over $\mathbf{Q}_p(\mu_{p^\infty})$. Since $D_{\mathrm{Sen}}(-)_{(-)}$ is exact we have

$$0 \to D_{\mathrm{Sen}}(D)_{(k)} \to D_{\mathrm{Sen}}(\widetilde{D})_{(k)} \to D_{\mathrm{Sen}}(D)_{(k)} \to 0. \tag{3.5}$$

Then, by definition we have that $\widetilde{D}$ has k as a Hodge-Tate-Sen weight if and only if (3.5) is split. Thus,

$$\mathfrak{t}_D^k = \ker\left(H^1(\mathrm{ad}\, D) \to \mathrm{Ext}^1_{\mathbf{Q}_p(\mu_{p^\infty})[\Gamma]}\left(D_{\mathrm{Sen}}(D)_{(k)}, D_{\mathrm{Sen}}(D)_{(k)} \right)\right)$$

The domain of the above arrow is easily seen to be one-dimensional.

Suppose now that D is a de Rham (φ, Γ)-module. Then, we define $\mathfrak{X}_{D,g} \subset \mathfrak{X}_D$ as the subset

$$\mathfrak{X}_{D,g}(A) = \left\{ D_A \in \mathfrak{X}_D(A) \colon D_A \text{ is de Rham} \right\}.$$

Notice that this actually defines a subfunctor. Further, it is closed under direct sum and we remarked in Lemma 2.11 that it was closed under subobjects. Similarly, if D is crystalline then we have a subfunctor

$$\mathfrak{X}_{D,f}(A) = \{D_A \in \mathfrak{X}_D(A) \colon D_A \text{ is crystalline}\}.$$

Just as before, this is also a functor which is closed under subobjects and direct sums in the sense above. It follows from Proposition 3.4 that the functors $\mathfrak{X}_{D,g}$ and $\mathfrak{X}_{D,f}$ define relatively representable subfunctors of $\mathfrak{X}_D$ (whenever they are defined).

In particular, we can define their Zariski tangent spaces $\mathfrak{t}_{D,g}$ and $\mathfrak{t}_{D,f}$. It follows from Proposition 2.17 that we have isomorphisms

$$\mathfrak{t}_{D,g} \cong H^1_g(\operatorname{ad} D), \text{ and}$$
$$\mathfrak{t}_{D,f} \cong H^1_f(\operatorname{ad} D),$$

at least when they are defined. The following computes the exact size of these tangent spaces.

Proposition 3.5. *Suppose that D is crystalline with Hodge-Tate weights $k_1, \dots, k_n$ (possibly with multiplicty). Then,*

$$\dim_L \mathfrak{t}_{D,f} = \dim_L \operatorname{End}_{(\varphi,\Gamma)}(D) + \#\{(i,j) \colon k_i < k_j\}$$

and

$$\dim_L \mathfrak{t}_{D,g} = \dim \mathfrak{t}_{D,f} + \dim D_{\mathrm{cris}}(D)^{\varphi=p} + \dim D_{\mathrm{cris}}(D)^{\varphi=p^{-1}}.$$

Proof. All of these follow from Proposition 2.19 applied to $\operatorname{ad} D = D \otimes_{\mathcal{R}_L} D^\vee$. □

The case we will most often is the following corollary.

Corollary 3.6. *If D is crystalline as a (φ,Γ)-module with distinct Hodge-Tate weights then*

$$\dim \mathfrak{t}_{D,f} = \dim_L \operatorname{End}_{(\varphi,\Gamma)}(D) + \frac{n(n-1)}{2}.$$

Proof. Since the Hodge-Tate weights are distinct exactly $\binom{n}{2}$ of the differences $k_i - k_j$ will be negative. Thus, we conclude from Proposition 3.5. □

3.2.2. Paraboline deformations. We now go on to explain the theory of paraboline deformations. Recall in §2.3 we described what it means to have a parabolization of a (φ,Γ)-module D_A over an element A in $\mathfrak{AR}_L$. Let D be a (φ,Γ)-module over $\mathcal{R}_L$ and P a parabolization of D.

Definition. *If D_A is a deformation of D to A then we say that D_A is a paraboline deformation with respect to P provided there is a parabolization P_A of D_A such that the isomorphism $\pi : D_A \otimes_A L \rightarrow D$ induces an isomorphism $P_{A,i} \otimes_A L \cong P_i$ for each i.*

The paraboline deformations define a functor $\mathfrak{X}_{D,P} : \mathfrak{AR}_L \rightarrow \underline{\mathrm{Set}}$ on points by

$$\mathfrak{X}_{D,P}(A) = \{D_A \in \mathfrak{X}_D(A) \colon D_A \text{ is a paraboline deformation with respect to } P\}.$$

Note that it is a functor. However, it is not clear (or true in general) that it is a subfunctor. Even in the case that it is a subfunctor, there is still work to show that it is relatively representable. These are our two mains goals, but first we give some examples.

Example 3.7. Let P to be the trivial parabolization $0 \subsetneq D$. Then we clearly have that $\mathfrak{X}_{D,P} = \mathfrak{X}_D$.

Example 3.8. Let P be a triangulation. The study of the deformation functors $\mathfrak{X}_{D,P}$ was one of the main topics of [**5**]. The authors of *loc. cit.* showed that trianguline deformations naturally appear infinitesimally in p-adic families. We will show in Theorem 4.13 and §4.3.1 that p-adic families carry large open loci of trianguline deformations, recovering the Bellaïche-Chenevier result.

The more general paraboline deformations, and many of the computations below, are discussed in [**16**]. However we include the proofs of many results since we are implicitly working towards critical triangulations and thus our hypotheses are slightly different. Before continuing, we need to set up some notation.

Given a parabolization P of length s, we set

$$\mathrm{End}_P(D) = \{f \in \mathrm{End}_{\mathcal{R}_L}(D)\colon f(P_j) \subset P_j \text{ for } j = 1, \dots, s\}.$$

Then, this is a (φ,Γ)-submodule and a direct summand of $\mathrm{End}_{\mathcal{R}_L}(D)$ as an $\mathcal{R}_L$-module. Note that

$$H^0(\mathrm{End}_P(D)) = \left\{f \in \mathrm{End}_{(\varphi,\Gamma)}(D)\colon f(P_j) \subset P_j \text{ for } j = 1, \dots, s\right\}.$$

We will always use the notation $\mathrm{Gr}_j P$ to denote the associated graded P_j/P_{j-1}. The answer to the question of $\mathfrak{X}_{D,P}$ being a subfunctor is given by the following result (see [**5**, Proposition 2.3.6] as well).

Lemma 3.9. *Assume that* $\mathrm{Hom}_{(\varphi,\Gamma)}(\mathrm{Gr}_j P, D/P_j) = (0)$ *for* $j = 1, \dots, s$. *Then,* $\mathfrak{X}_{D,P}$ *is a subfunctor of* $\mathfrak{X}_D$.

Proof. By induction on the length of P, we must show that if D_A is a paraboline deformation with respect to P, with parabolization P_A, then $P_{A,1}$ is uniquely determined as a (φ,Γ)-submodule of D_A. Let $m := \mathrm{rank}\, P_1$ and suppose that $\widetilde{P}_1$ is a rank m saturated (φ,Γ)-submodule of D_A deforming P_1. We claim that

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\widetilde{P}_1, D_A/P_{A,1}) = (0).$$

If so, we see that $\widetilde{P}_1 \subset P_{A,1}$. Since each of $\widetilde{P}_1$ and $P_{A,1}$ are saturated inside D_A of the same rank, we are done.

To prove the claim, we begin with the hypothesis that $\mathrm{Hom}_{(\varphi,\Gamma)}(P_1, D/P_1) = (0)$. Since $D_A/P_{A,1}$ is a successive extension of D/P_1 by itself we see easily, by the left exactness of $\mathrm{Hom}_{(\varphi,\Gamma)}(P_1,-)$, that

$$\mathrm{Hom}_{(\varphi,\Gamma)}(P_1, D_A/P_{A,1}) = (0).$$

Now apply the same argument to the first coordinate: $\widetilde{P}_1$ is a successive extension of P_1 by itself and so left-exactness of $\mathrm{Hom}_{(\varphi,\Gamma)}(-, D_A/P_{A,1})$ finishes the claim. ☐

Example 3.10. Assume that P is a triangulation with parameter $(\delta_1, \dots, \delta_n)$ such that $\delta_i^{-1}\delta_j \notin \widehat{S}^+$ if $i < j$ (this is the hypothesis in [**5**, Proposition 2.3.6]). In that case, if P' is any subparabolization of P then the hypothesis of Lemma 3.9 is true for P'. For example, we could apply this to $P' = P^{\mathrm{nc}}$.

Remark. The condition in Lemma 3.9 is obviously necessary. Indeed, suppose that $D = \mathcal{R}_L(\delta)^{\oplus 2}$ with basis $\mathbf{e}_1$ and $\mathbf{e}_2$ giving a triangulation $0 \subsetneq \mathcal{R}_L\mathbf{e}_1 \subsetneq D$. Then, if we consider the constant deformation $D \otimes_L L[\varepsilon]$, we have many triangulations deforming the one downstairs. For example, we have the two $0 \subsetneq \mathcal{R}_{L[\varepsilon]}(\mathbf{e}_1) \subsetneq D_{L[\varepsilon]}$ and $0 \subsetneq \mathcal{R}_{L[\varepsilon]}(\mathbf{e}_1 + \varepsilon\mathbf{e}_2) \subsetneq D_{L[\varepsilon]}$.

We now moves towards the relative representability of $\mathfrak{X}_{D,P}$. Our method of proving that $\mathfrak{X}_{D,P}$ is relatively representable is to make use of the criterion in Proposition 3.4. In order to explain the validity of point (iii) in *loc. cit.* however, we have to make a short detour into a discussion of irreducible (φ,Γ)-modules. The final proof of relative representability will be given in Theorem 3.25.

Recall that by Proposition 1.7 a finitely generated module D over $\mathcal{R}_L$ is free if and only if it is torsion-free. We will use this constantly without further comment.

Definition. *A* (φ,Γ)*-module* D *over* $\mathcal{R}_L$ *is irreducible if the only proper, saturated, submodule is the zero module.*

Lemma 3.11. *Let D be a (φ,Γ)-module over $\mathcal{R}_L$. The following are equivalent:*

(a) D is irreducible;

(b) for every submodule $D_0 \subset D$ either $D_0 = (0)$ or D/D_0 is a torsion (φ,Γ)-module;

(c) for every submodule $D_0 \subset D$ either $D_0 = (0)$ or there exists an $n \gg 0$ such that $t^n D \subset D_0$.

Proof. The equivalence of (a) and (b) is the definition of irreducible and saturated. The equivalence of (b) and (c) follows from Corollary 2.8. □

Throughout the rest of the section we use D to denote a (φ,Γ)-module and we will prefer to use π to denote an irreducible (φ,Γ)-module. We have the following version of Schur's lemma.

Lemma 3.12. *Suppose that π and π' are two irreducible (φ,Γ)-modules. If $\alpha : \pi \to \pi'$ is a morphism then either $\alpha = 0$ or α is injective and $\alpha[1/t]$ is an isomorphism.*

Proof. If α is not injective then $\ker\alpha$ is a non-zero saturated submodule of π. By the irreduciblity of π we deduce that $\alpha = 0$. Now suppose that α is injective. In that case, π is a (φ,Γ)-module of π' and the irreducibility of π' implies that π/π' is t-torsion by Lemma 3.11 and the result now follows. □

Lemma 3.13. *Suppose that π is irreducible and $\pi' \subset \pi$ is any submodule. Then, π' is also irreducible.*

Proof. If $\pi'' \subset \pi'$ then we have $\pi'/\pi'' \subset \pi/\pi''$. As π is irreducible, π/π'' is pure t-torsion and thus so is π'/π''. So, the result follows from Lemma 3.11. □

Example 3.14. If $\delta : \mathbf{Q}_p^\times \to L^\times$ be a continuous character then the rank one (φ,Γ)-module $\pi = \mathcal{R}_L(\delta)$ is irreducible. Indeed, it follows from Proposition 2.7 that the only submodules are of the form $t^r\mathcal{R}_L(\delta)$ with $r \geq 0$.

The previous results and the rank one example leads us to give the following equivalence relation on irreducible modules.

Definition. *Let π and π' be irreducible (φ,Γ)-modules. We say that π and π' are equivalent, and write $\pi \sim \pi'$, if the (φ,Γ)-modules $\pi[1/t]$ and $\pi'[1/t]$ over $\mathcal{R}_L[1/t]$ are isomorphic.*

Notice that this is clearly an equivalence relation on the set of irreducible (φ,Γ)-modules. The $\mathcal{R}_L$-rank is constant on each equivalence class.

Lemma 3.15. *Given two irreducible (φ,Γ)-modules π and π, we have $\pi \sim \pi'$ if and only if there exists integers $r \geq s \in \mathbf{Z}$ and inclusions $t^r\pi \hookrightarrow \pi' \hookrightarrow t^s\pi$ whose composition $t^r\pi \hookrightarrow t^s\pi$ is the identity.*

Proof. This follows immediately from the definition and the fact that π (resp. π') is a finitely generated $\mathcal{R}_L$-submodule of $\pi[1/t]$ (resp. $\pi'[1/t]$). □

Lemma 3.16. *Suppose that $\pi \subset D$ is a submodule. Then, π is irreducible if and only if its saturation π^{sat} is irreducible. In particular, π is equivalent to an irreducible, saturated, submodule of D.*

Proof. If π^{sat} is irreducible then so is π by Lemma 3.13. Suppose that π is irreducible and that $\pi' \subset \pi^{\mathrm{sat}}$ is a proper, saturated, submodule. By definition of the saturation, we can choose an $r \gg 0$ such that $t^r\pi^{\mathrm{sat}} \subset \pi$. Since $t^r\pi^{\mathrm{sat}}$ is irreducible (by Lemma 3.13 again) it follows that $t^r\pi' = (0)$ and thus $\pi' = (0)$. □

Definition. *Let D be a (φ,Γ)-module. We say that an irreducible (φ,Γ)-module π is an irreducible constituent of D if there exists a (φ,Γ)-submodule $D' \subset D$ and a surjection $D' \twoheadrightarrow \pi$.*

Notice that we don't *a priori* require D' to be a saturated submodule.

Proposition 3.17. *The following are equivalent.*

(a) π *is an irreducible constituent of* D.
(b) There exists a saturated submodule $D' \subset D$ *and a surjection* $D' \twoheadrightarrow \pi'$ *for some irreducible* (φ, Γ)*-module* π' *which is equivalent to* π.
(c) There exists a quotient $D \twoheadrightarrow D''$ *of* (φ, Γ)*-modules such that* $\pi \hookrightarrow D''$.
(d) There exists a quotient $D \twoheadrightarrow D''$ *of* (φ, Γ)*-modules and an irreducible submodule* $\pi' \hookrightarrow D''$ *which is saturated and* π' *is equivalent to* π.

Proof. Starting at the bottom, (d) is equivalent to (c) by Lemma 3.16 (take π' to be the saturation of π inside D''). Moving to the top, obviously (b) implies (a). To see the reverse implication we choose $D' \subset D$ with π being a quotient of D'. Consider $(D')^{\mathrm{sat}} \subset D$. Notice that the map $D' \to \pi$ defines a map $(D')^{\mathrm{sat}} \to \pi[1/t]$. Since $(D')^{\mathrm{sat}}$ is a (φ, Γ)-module, the image π' is a (φ, Γ)-module necessarily equivalent to π by Lemma 3.15.

It remains to prove (b) is equivalent to (d). First assume (b) and let $D' \subset D$ be a saturated submodule such that π' is a quotient D'. We set $D'_0 := \ker(D' \to \pi')$. Since D'_0 is saturated inside D' and D' is saturated inside D'_0, the quotient $D'' := D/D'_0$ is a (φ, Γ)-module. Moreover,

$$\pi' \cong D'/D'_0 \subset D/D'_0$$

is saturated, as the quotient is D/D'. Thus π' is a saturated submodule of the quotient D'' of D, which shows (d).

For the reverse, let us choose D'' and π' as given in (d). Write D' for

$$D' := \ker\left(D \to D''/\pi\right).$$

Since π' is saturated inside D'' we have that D' is a (φ, Γ)-submodule of D and thus free. We write as well $D'_0 := \ker(D \to D'')$. Then,

$$\pi' \cong D'/D'_0 \hookrightarrow D/D'_0 \cong D''.$$

Thus π' is a quotient of the saturated submodule D', which shows (b). □

Remark. We note that while conditions (b) and (d) seem more natural, for flexibility and clarity in our arguments we will often use the non-saturated versions (a) and (c) above.

Example 3.18. The irreducible constituents of $\mathcal{R}_L$ are the (φ, Γ)-modules of the form $t^r \mathcal{R}_L$ with $r \geq 0$. Thus, irreducible consituents are not unique, even for an irreducible (φ, Γ)-module.

We now consider the following situation. Let $\Pi = \{\pi_1, \ldots, \pi_s\}$ be a list of irreducible (φ, Γ)-modules. If $\pi_i = \mathcal{R}_L(\delta_i)$ is a character for each i then we will write $\Pi = \{\delta_1, \ldots, \delta_s\}$.

Definition. *A* (φ, Γ)*-module* D *is of type* Π *if for every irreducible constituent* π *of* D *there exists a* $\pi_i \in \Pi$ *such that* $\pi \sim \pi_i$.

Example 3.19. Let $\Pi = \{\delta\}$. The rank one modules of type Π are all the modules $\mathcal{R}_L(z^r\delta)$ with $r \in \mathbf{Z}$.

Proposition 3.20. *Let* D *be a* (φ, Γ)*-module.*

(a) If D_1 *and* D_2 *are submodules of* D *of type* Π *then so is* $D_1 + D_2 \subset D$.
(b) If $f : D \to E$ *is a map of* (φ, Γ)*-modules and* $D' \subset D$ *is of type* Π *then so is* $E' = f(D')$.
(c) If D *has type* Π *and* $D' \subset D$ *then* D' *has type* Π *as well. If* D/D' *is torsion then the converse holds.*
(d) If $0 \to D' \to D \to D'' \to 0$ *is a short exact sequence and* D' *and* D'' *are each of type* Π *then so is* D.

Proof. We first prove (a). Write D_3 for the sum $D_1 + D_2$. Note that it is still a (φ, Γ)-module since it is finitely generated and torsion-free (being a submodule of D). Suppose that π is an irreducible constituent of D_3. Choose a (φ, Γ)-submodule $N_3 \subset D_3$ and a surjection $\alpha : N_3 \twoheadrightarrow \pi$. Consider $N_1 := N_3 \cap D_1 \subset D_1$. Since π is irreducible, the image $\alpha(N_1) \subset \pi$ is either zero or an irreducible module $\pi' \sim \pi$. If we are in the latter case, then by definition π' is an irreducible constituent of D_1 and thus $\pi' \sim \pi_i$ for some i, since D_1 is of type Π. Thus $\pi \sim \pi_i$.

Now suppose that $\alpha(N_1) = (0)$. If that is the case then we consider a sequence of *generalized* (φ, Γ)*-modules*

$$E_1 := N_3/N_1 \cong N_3 + D_1/D_1 \hookrightarrow D/D_1 \cong D_2/D_2 \cap D_1 =: E_2.$$

Recall we use the notation $(-)_{\mathrm{tor}}$ to denote $\mathcal{R}_L$-torsion submodules. By assumption, π is a quotient of the submodule $E_1 \subset E_2$. Moreover, though neither of these are necessarily torsion-free we do know that $\alpha(E_{1,\mathrm{tor}}) = (0)$ since π is torsion-free. Thus π is a quotient of $E_{1,\mathrm{free}}$. Moreover, $E_{1,\mathrm{tor}} = E_{2,\mathrm{tor}} \cap E_1$ and thus we have a natural inclusion $E_{1,\mathrm{free}} \subset E_{2,\mathrm{free}}$. Thus, π is a quotient of a (φ, Γ)-submodule $E_{1,\mathrm{free}}$ of the (φ, Γ)-module $E_{2,\mathrm{free}}$. Since $E_{2,\mathrm{free}}$ is itself a quotient of the (φ, Γ)-module D_2 we have the π is an irreducible constituent of D_2. Since D_2 is of type π, there exists an i such that $\pi \sim \pi_i$.

Dispensing with (b) is easy. Let $E' := f(D')$ and suppose that π is an irreducible submodule of a quotient $E' \twoheadrightarrow E''$. Then, π is an irreducible submodule of a quotient $D' \twoheadrightarrow E' \twoheadrightarrow E''$ and thus $\pi \sim \pi_i$ for some i, since D' has type Π.

Consider part (c). Suppose that π is an irreducible constituent of D'. Thus there is a submodule $D'' \subset D'$ such that π is a quotient of D''. Since D'' is also a submodule of D and D is of type Π we have that $\pi \sim \pi_i$ for some i. As π was arbitrary, D' has type Π. In the case that D/D' is torsion, $(D')^{\mathrm{sat}} = D$ and there exists an $r \gg 0$ such that $t^r D \subset D'$. By the first implication of this part, $t^r D$ is of type Π. Then, if π is a constituent of D we get that $t^r \pi$ is a constituent of $t^r D$. Since $\pi \sim t^r \pi$ the result follows from $t^r D$ being of type Π.

Part (d) remains. Suppose that π is a constituent of D and choose $Q \subset D$ such that π is a quotient of Q. If the natural map $\beta : Q \cap D' \to \pi$ is non-zero then $\pi \sim \pi_i$ for some i as D' has type Π. If β is zero then we consider the free module

$$Q/Q \cap D' \cong Q + D'/D' \subset D/D' \cong D''.$$

Since π is a quotient of the submodule, it is a constintuent of D'' and so again we get that $\pi \sim \pi_i$ for some i. $\square$

We now move on to give our main examples.

Definition. *Let* $\Pi = \{\pi_1, \ldots, \pi_s\}$ *be a list of irreducible* (φ, Γ)*-modules and let* D *be any* (φ, Γ)*-module. We define* D_Π *to be the largest* (φ, Γ)*-submodule of* D *such that* D_Π *is of type* Π.

Lemma 3.21. *The association* $F_\Pi(D) := D_\Pi$ *defines a left exact functor on* (φ, Γ)*-modules over* $\mathcal{R}_L$.

Proof. First, there exists a largest submodule D_Π by part (a) above. Thus F_Π is well-defined. Second, F_Π is a functor by part (b) above. Indeed, if $f : D \to E$ is a map then $f(D_\Pi)$ is of type Π and thus $f(D_\Pi) \subset E_\Pi$, by maximality of E_Π inside E.

It remains to check the exactness. Suppose that we have a SES $0 \to D' \to D \xrightarrow{g} D'' \to 0$ and consider the induced sequence of maps

$$0 \to D'_\Pi \to D_\Pi \xrightarrow{g} D''_\Pi.$$

It is obviously exact on the left. To show that it is exact in the middle we have to show that $\ker(g\big|_{D_\Pi}) \subset D'_\Pi$ (the other containment being trivial). However, $\ker(g\big|_{D_\Pi}) \subset D_\Pi \cap D'$ is a submodule of D' of type Π (by part (c) above) and thus this follows from maximality. $\square$

Throughout the rest of this subsction, we use Π to denote a list of irreducible (φ,Γ)-modules $\Pi = \{\pi_1,\dots,\pi_s\}$.

Proposition 3.22. *Let D be a (φ,Γ)-module. Then $D_\Pi \subset D$ is saturated as a $\mathcal{R}_L$-module.*

Proof. Let $D' := (D_\Pi)^{\mathrm{sat}}$. Since D'/D_Π is torsion, D' is of type Π by Proposition 3.20(c). By the maximaility of D_Π we deduce that $D' = D_\Pi$. $\square$

Now that we have gotten through our preliminaries on irreducible (φ,Γ)-modules and the notion of having a type, we will work to prove the relative representability of paraboline deformation functors.

Proposition 3.23. *Suppose that π is an irreducible (φ,Γ)-module. Then,*

$$F_\Pi(\pi) = \begin{cases} \pi & \textit{if } \pi \sim \pi' \textit{ for some } \pi' \in \Pi \\ 0 & \textit{othwerise} \end{cases}$$

Proof. By Proposition 3.22 we have that $F_\Pi(\pi)$ is saturated inside π. As π is irreducible, we have that either $F_\Pi(\pi)$ is zero or π itself. Further, since π is an irreducible constituent of itself, we clearly have π is of type Π if and only if π is equivalent to a member of Π. $\square$

Example 3.24. Recall that $\widehat{S}^+ = \left\{z^{-j} \colon j \geq 0\right\}$. $F_{\{\eta\}}(\mathcal{R}_L(\delta)) = \mathcal{R}_L(\delta)$ if and only if $(\delta\eta^{-1})^{\pm 1} \in \widehat{S}^+$.

Let D be a (φ,Γ)-module. If D is irreducible then it is of type $\{D\}$. Otherwise, there exists a non-zero saturated submodule $D_0 \subset D$. By induction on $\operatorname{rank}_{\mathcal{R}_L} D$ we have that D_0 contains a non-zero irreducible submodule π. Furthermore, by Lemma 3.16 we can assume that π is saturated inside D_0, and thus in D as well. By induction on the rank of D again we deduce that there exists a finite list $\Pi_D = \{\pi_1,\dots,\pi_s\}$ of irreducible (φ,Γ)-modules such that D is a successive extension of π_i by π_j. Notice that the individuals members of the list are neither unique nor is the ordering of a given list of irreducible constituents. However, since D is of type Π_D any two ways of generating the list Π_D (in an apparent minimal sense) are the same up to reordering and $\sim$. Finally, notice that if D has the structure of a module over a commutative ring A then $F_\Pi(D)$ inherits a natural A-module structure as well.

Theorem 3.25. *Let D be a trianguline (φ,Γ)-module with triangulation P' and parameter $(\delta_1,\dots,\delta_n)$ such that if $i \neq j$ then $\delta_i^{-1}\delta_j \notin \widehat{S}^+$. Let P be any subparabolization of P'. Then,*

- $\mathrm{Hom}_{(\varphi,\Gamma)}(\mathrm{Gr}_j\, P, D/P_j) = (0)$ *for $j = 1,\dots,s$ and*
- $\mathfrak{X}_{D,P}$ *is a relatively representable subfunctor of $\mathfrak{X}_D$.*

Proof. The hypothesis on the parameter δ_i implies that

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta_j)) = (0)$$

if $i \neq j$. It is easy to check by induction and dévissage that the first point is true. In particular, Lemma 3.9 implies that $\mathfrak{X}_{D,P}$ is a subfunctor of $\mathfrak{X}_D$. In order to prove that $\mathfrak{X}_{D,P}$ is relatively representable we will use Proposition 3.4. The first condition is clear: if $D_A \in \mathfrak{X}_{D,P}(A)$ then $P_A \otimes_A A'$ is a parabolization of $D_{A'}$ deforming P.

The second condition is explained in [**16**, Proposition 3.4] but we include the proof here for convenience. Choose a basis $\mathscr{B}$ adapted to the triangulation P'. We have to show that if $A, B \in \mathfrak{AR}_L$ then the diagram

$$\begin{array}{ccc} \mathfrak{X}_{D,P}(A \times_L B) & \longrightarrow & \mathfrak{X}_{D,P}(A) \times \mathfrak{X}_{D,P}(B) \\ \downarrow & & \downarrow \\ \mathfrak{X}_D(A \times_L B) & \longrightarrow & \mathfrak{X}_D(A) \times \mathfrak{X}_D(B) \end{array}$$

is cartesian (recall that $\mathfrak{X}_D(L) = \{D\}$ is a point). So, suppose that $D_{A\times_L B}$ is a deformation of D such that D_A and D_B are both paraboline with respect to P. If S is any of the rings $A \times_L B$, A or B then $D_S \cong D \otimes_L S$ as a $\mathcal{R}_S$-module and so we view $\mathscr{B}$ simultaneously as a $\mathcal{R}_S$-basis of D_S. Thus we may consider the matrices $\{[\varphi_{D_S}]_{\mathscr{B}}, [\gamma_{D_S}]_{\mathscr{B}}\}$. We use $*$ to denote either φ or γ. We have that $[*_{D_{A\times_L B}}]_{\mathscr{B}} = ([*_{D_A}]_{\mathscr{B}}, [*_{D_B}]_{\mathscr{B}})$ inside

$$\mathrm{GL}_n(\mathcal{R}_{A\times_L B}) \cong \mathrm{GL}_n(\mathcal{R}_A) \times_{\mathrm{GL}_n(\mathcal{R}_L)} \mathrm{GL}_n(\mathcal{R}_B). \tag{3.6}$$

Denote by $R(S)$ (this is the notation of *loc. cit.*) the elements of $\mathrm{GL}_n(\mathcal{R}_S)$ which are block upper triangular with block sizes given in order by the parabolization P. It is evident from (3.6) that $R(A\times_L B) = R(A)\times_{R(L)} R(B)$. Denote as well $R^1(S) = \ker(R(S) \to R(L))$. Now, what we are told (by asking that D_A and D_B are paraboline) is that there are elements $r_A \in R^1(A)$ and $r_B \in R^1(B)$ such that $[*_{D_A}]_{r_A\mathscr{B}}$ and $[*_{D_B}]_{r_B\mathscr{B}}$ are in $R(A) \times R(B)$. Since $r_A \bmod \mathfrak{m}_A = r_B \bmod \mathfrak{m}_B$ (both are the identity) we get that the pair (r_A, r_B) defines an element of $r_{A\times_L B} \in R^1(A \times_L B)$. Moreover, we have that $[*_{D_{A\times_L B}}]_{r_{A\times_L B}\mathscr{B}} \in R(A \times_L B)$ and thus $D_{A\times_L B}$ is a paraboline deformation.

We now move on to the third condition of Proposition 3.4. We need to assume that $A \subset A'$, that D_A is a deformation of D to A and that $D_{A'} := D_A \otimes_A A'$ is a paraboline deformation of D with respect to P. Our conclusion should be that D_A was paraboline already. It suffices by induction on the rank of D to just treat the case of a single step $0 \subsetneq P_i \subsetneq D$. Consider the set $\Pi = \{\delta_1, \dots, \delta_i\}$ of irreducible constituents for P_i. By our assumption on the parameter and Example 3.24 we have that for each $j = i+1, \dots, n$ that $F_\Pi(\delta_j) = (0)$. Thus, by the left exactness of F_Π we have $F_\Pi(D/P_i) = (0)$ and $P_i = F_\Pi(P_i) = F_\Pi(D)$.

Now consider the deformation D_A and its constant scalar extension $D_{A'} = D_A \otimes_A A'$. By assumption we have that there exists a saturated (φ,Γ)-submodule $P_{i,A'} \subset D_{A'}$ deforming P_i. By the left exactness of F_Π we have that $F_\Pi(D_{A'}/P_{i,A'}) = (0)$ and $P_{i,A'} = F_\Pi(P_{i,A'}) = F_\Pi(D_{A'})$ is free over $\mathcal{R}_{A'}$. Furthermore, if M is any finite length A-module then $D_A \otimes_A M$ is a (φ,Γ)-module over $\mathcal{R}_L$ (as M is a vector space over L) and

$$\mathrm{rank}_{\mathcal{R}_L} F_\Pi(D_A \otimes_A M) \leq i \,\mathrm{len}_A(M). \tag{3.7}$$

This follows from considering a composition series for M as an A-module and the left exactness of F_Π. Consider the short exact sequence

$$0 \to F_\Pi(D_A) \to F_\Pi(D_{A'}) \to F_\Pi(D_A \otimes_A A'/A).$$

The middle term is a (φ,Γ)-module of rank exactly $\mathrm{rank}_{\mathcal{R}_L} P_{i,A'} = i \,\mathrm{len}_A A'$ and the final term has rank at most $i(\mathrm{len}_A A' - \mathrm{len}_A A)$ by (3.7). Thus $F_\Pi(D_A)$ has rank at least $i \,\mathrm{len}_A A$. Applying (3.7) for an upper bound, we get that $F_\Pi(D_A)$ is a (φ,Γ)-module over $\mathcal{R}_L$ of rank $i \,\mathrm{len}_A A$.

Next, Proposition 3.22 implies that since $F_\Pi(D_A)$ is saturated as a $\mathcal{R}_L$-module inside D_A. Thus, the same must be true of the image of $F_\Pi(D_A) \otimes_A L$ inside $D_A \otimes_A L = D$. Indeed, $D_A/F_\Pi(D_A)$ is a successive extension of $\mathrm{coker}(F_\Pi(D_A) \otimes_A L \to D)$ by itself and taking torsion is left exact. Since $\mathrm{rank}_{\mathcal{R}_L} F_\Pi(D_A) \otimes_A L = i$ we deduce that the image of $F_\Pi(D_A) \otimes_A L$ inside D is a saturated (φ,Γ)-submodule of rank i and type Π; it follows from this that the natural map $F_\Pi(D_A) \to P_i$

is surjective. It must also be injective as the two objects have the same rank over $\mathcal{R}_L$. Thus, $F_\Pi(D_A)\otimes_A L \cong P_i$. Finally, we consider the exact sequence

$$0 \to F_\Pi(D_A) \to D_A \to D_A/F_\Pi(D_A) \to 0.$$

As the sequence remains exact over the residue field L (by what we just said) we deduce that

$$\mathrm{Tor}_1^A(D_A/F_\Pi(D_A), L) = \mathrm{Tor}_1^A(F_\Pi(D_A), L) = (0)$$

and everything in sight is free over A. Finally by [**5**, Lemma 2.2.3(ii)] we deduce that $F_\Pi(D_A)$ is a (φ,Γ)-module over $\mathcal{R}_A$ (as it free of finite rank over $\mathcal{R}_L$ and free over A). Thus D_A is a paraboline deformation of D with respect to $P_i \subset D$. □

Corollary 3.26. *Suppose that D is a crystalline (φ,Γ)-module with distinct crystalline eigenvalues. Then, for any parabolization P of D we have that $\mathfrak{X}_{D,P} \subset \mathfrak{X}_D$ is a relatively representable subfunctor.*

Proof. By Proposition 2.21 any parabolization of a crystalline (φ,Γ)-module can be realized as a subparabolization of a triangulation. Thus Theorem 3.25 implies that it is enough to show that for any triangulation P' with parameter $(\delta_1,\dots,\delta_n)$, we have $\delta_i\delta_j^{-1} \notin \widehat{S}^+$.

Let P' be any triangulation of D. The dictionary of Proposition 2.21 tells us that the parameter is given by $\delta_i = z^{-s_i}\,\mathrm{unr}(\phi_i)$ for some orderings $(\phi_1,\dots,\phi_n)$ of the crystalline eigenvalues and $(s_1,\dots,s_n)$ of the Hodge-Tate weights. Thus $\delta_i\delta_j^{-1} = z^{s_j-s_i}\,\mathrm{unr}(\phi_i\phi_j^{-1}) \notin \widehat{S}^+$ unless $\phi_i = \phi_j$. □

Remark. If $P^+ \subset D$ is a saturated (φ,Γ)-submodule such that $F_{\Pi_{P^+}}(D/P^+) = (0)$ then the same proof will show that $\mathfrak{X}_{D,P^+} \to \mathfrak{X}_D$ satisfies Proposition 3.4 as well.

We now have, under a regularity hypothesis, a tangent space $\mathfrak{t}_{D,P} \subset \mathfrak{t}_D$ whose dimension we will compute. Recall that we previously defined the $\mathcal{R}_L$-module $\mathrm{End}_P(D)$. Choosing a basis of D, adapated to the filtration P, we realize any element of $\mathrm{End}_P(D)$ as a block upper triangular matrix in $M_n(\mathcal{R}_L)$ (the blocks being the successive ranks of each step P_i in P). In particular, we see easily then that

$$\mathrm{rank}_{\mathcal{R}_L}\mathrm{End}_P(D) = \sum_{i\leq j} n_in_j, \qquad n_i := \mathrm{rank}_{\mathcal{R}_L} P_i. \tag{3.8}$$

On the other hand, under the identification(s) (3.4) one has (see [**16**, Proposition 3.6]) that there is a commuting diagram

$$\begin{array}{ccc} \mathfrak{t}_D & \xrightarrow{\cong} & H^1(\mathrm{End}_{\mathcal{R}_L}(D)) \\ \uparrow & & \uparrow \\ \mathfrak{t}_{D,P} & \xrightarrow[\cong]{} & H^1(\mathrm{End}_P(D)) \end{array}.$$

This gives us the following formula.

Proposition 3.27. *Assume that D is a crystalline (φ,Γ)-module with distinct crystalline eigenvalues. Let P be a parabolization of D and let $n_i = \mathrm{rank}_{\mathcal{R}_L} P_i$. Then,*

$$\dim_L \mathfrak{t}_{D,P} = \sum_{i\leq j} n_in_j + \dim_L \mathrm{End}_{(\varphi,\Gamma)}(D) + \dim_L H^2(\mathrm{End}_P(D)).$$

Proof. Notice that if we replace $\mathrm{End}_{(\varphi,\Gamma)}(D)$ by $\mathrm{End}_{(\varphi,\Gamma),P}(D)$ then this is a consequence of the Euler-Poincaré-Tate characteristic formula, Proposition 2.16 and (3.8). Thus, it suffices to show that $\mathrm{End}_{(\varphi,\Gamma),P}(D) = \mathrm{End}_{(\varphi,\Gamma)}(D)$, i.e. we need to prove that if $f : D \to D$ is (φ,Γ)-equivariant then $f(P_j) \subset P_j$ for all j. We prove this by induction on j. Notice that by assumption on D we have that $\mathrm{Hom}_{(\varphi,\Gamma)}(\mathrm{Gr}_j P, D/P_j) = (0)$.

If $j = 1$ then by hypothesis we have that composite

$$P_1 \xrightarrow{f|_{P_1}} D \to D/P_1$$

must be the zero map. Thus $f(P_1) \subset P_1$. Suppose that we have shown the claim for $j-1$. In that case, f necessarily defines a well-define (φ, Γ)-equivariant map $\overline{f} : D/P_{j-1} \to D/P_{j-1}$. Just as in the previous case, the composite

$$\mathrm{Gr}_j P \xrightarrow{\overline{f}|_{\mathrm{Gr}_j P}} D/P_{j-1} \twoheadrightarrow D/P_j$$

must be the zero map and thus $\overline{f}(\mathrm{Gr}_j P) \subset \mathrm{Gr}_j P$, which implies that $f(P_j) \subset P_j$. □

As an example of the vanishing of the final term in the above summation, we have the following result.

Proposition 3.28. *Suppose that D is a crystalline (φ, Γ)-module such that for any two eigenvalues ϕ and ϕ' of φ acting on $D_{\mathrm{cris}}(D)$ we have $\phi \neq p\phi'$. Then, for any parabolization P of D we have $H^2(\mathrm{End}_P(D)) = (0)$.*

Proof. Since $\mathrm{End}_P(D) \subset \mathrm{End}_{\mathcal{R}_L}(D)$ is a direct summand as a $\mathcal{R}_L$-module, the crystalline eigenvalues of $\mathrm{End}_P(D)$ are all of the form $\phi^{-1}\phi'$ for ϕ and ϕ' crystalline eigenvalues of $D_{\mathrm{cris}}(D)$.

Choose any triangulation of D (it is crystalline, so there are many) and let its parameter be $(\delta_1, \dots, \delta_n)$. Then, the hypothesis on the eigenvalues implies that $H^2(z^r \delta_i \delta_j^{-1}) = (0)$ for all i, j and $r \in \mathbf{Z}$. Since $\mathrm{End}_{\mathcal{R}_L}(D)$ is trianguline with a parameter $(\delta_i \delta_j^{-1})_{i,j}$, $\mathrm{End}_P(D)$ is trianguline as well, with a parameter of the form $(z^{r_{ij}} \delta_i \delta_j^{-1})$ with $r_{ij} \in \mathbf{Z}$. In any case, the first half of this paragraph implies that $H^2(\mathrm{End}_P(D)) = (0)$. □

We finish our discussion of paraboline deformations by pushing towards what more one can say about the interaction between paraboline deformations and other deformation functors. Here, for the first time in this section, we see that the role that non-criticality plays.

Proposition 3.29 ([**16**, Proposition 3.13]). *Suppose that D is a crystalline (φ, Γ)-module over $\mathcal{R}_L$. Suppose, moreover, that P is a non-critical parabolization of D and that $\mathrm{Hom}_{(\varphi,\Gamma)}(D/P_j, \mathrm{Gr}_j P) = (0)$ for all j. Then, $\mathfrak{X}_{D,f}$ is a subfunctor of $\mathfrak{X}_{D,P}$.*

We will come back to this point (see Theorem 3.38) but let us pause here and return to Proposition 3.27. Notice then that the term $\mathrm{End}_{(\varphi,\Gamma)}(D)$ appearing in $\mathfrak{t}_{D,P}$ also appears in the computation of $\mathfrak{t}_{D,f}$ (see Proposition 3.5). In particular, the quotient $\mathfrak{t}_{D,P}/\mathfrak{t}_{D,f}$ couldn't care less about the decomposability of D. On the other hand, paraboline deformations also interact with naked deformation spaces. Suppose that $D_A \in \mathfrak{X}_{D,P}(A)$ is a paraboline deformation with parabolization P_A. We can then consider the associated gradeds $\mathrm{Gr}_j P_A$. Writing this down is functorial in A and so we have a natural transformation

$$\mathfrak{X}_{D,P} \to \prod_{j=1}^{s} \mathfrak{X}_{\mathrm{Gr}_j P}. \tag{3.9}$$

Under some weak hypothesis, we will be able to exploit this map to reduce computations on D to computations on each associated graded.

Proposition 3.30 ([**16**, Proposition 3.7]). *Suppose that*

$$H^2(\mathrm{Hom}_{\mathcal{R}_L}(D/P_j, \mathrm{Gr}_j P)) = (0)$$

for $j = 1, \dots, s$. Then, for each $A \in \mathfrak{AR}_L$ we have that (3.9) is surjective: $\mathfrak{X}_{D,P}(A) \twoheadrightarrow \prod_{j=1}^{s} \mathfrak{X}_{\mathrm{Gr}_j P}(A)$.

In the situation above, we say that (3.9) is surjective on points. We end this section with just an example of when the hypothesis is satisfied.

EXAMPLE 3.31. Suppose that D is crystalline, with regular Hodge-Tate weights and such that $\phi'\phi^{-1} \neq p$ for each pair of distinct eigenvalues ϕ and ϕ' of φ acting on $D_{\mathrm{cris}}(D)$. Let P be any triangulation of D with parameter $(\delta_1, \ldots, \delta_n)$. We claim that the hypotheses of Proposition 3.30 holds.

For each j, we have

$$\mathrm{Hom}_{\mathcal{R}_L}(D/P_j, \mathrm{Gr}_j P) = (D/P_j)^\vee(\delta_j).$$

Since $(D/P_j)^\vee$ has a triangulation with parameter $(\delta_{j-1}^{-1}, \ldots, \delta_1^{-1})$, and H^2 is right exact, it suffices now to show that $\delta_i^{-1}\delta_j \notin \widehat{S}^-$ for $i = 1, \ldots, j-1$. However, if we have

$$z^{\mathrm{wt}(\delta_i)-\mathrm{wt}(\delta_j)}\,\mathrm{unr}(\phi_i^{-1}\phi_j) = \delta_i^{-1}\delta_j = z^m\,|z|$$

with $m \geq 1$ then we see right away that $m = \mathrm{wt}(\delta_i) - \mathrm{wt}(\delta_j)$ and $\phi_i = p\phi_j$, contradicting our assumption on the eigenvalues.

3.2.3. Deformations of Kisin-type. We've seen deformations arising from p-adic Hodge theory and deformations arising from the structure of (φ, Γ)-modules associated to Galois representations. The latter actually originally arose in [**5**] partly inspired by deformation theoretic computations of Kisin [**43**]. We now revisit the origins of this connection with a view towards the applications in Chapter 5. In particular, we will produce upper bounds in §3.2.4 for certain deformation rings contingent on computing the Kisin-type deformation rings described below.

For the moment, let D be a crystalline (φ, Γ)-module over $\mathcal{R}_L$ and assume that 0 is its least Hodge-Tate weight, with multiplicity one. Assume as well that ϕ is a crystalline eigenvalue of φ acting on $D_{\mathrm{cris}}(D)$. We define a deformation problem

$$\mathfrak{X}_D^\phi = \left\{ D_A \in \mathfrak{X}_D(A) \colon D_{\mathrm{cris}}^+(D_A)^{\varphi=\phi_A} \text{ is free of rank one for some } \phi_A \equiv \phi \bmod \mathfrak{m}_A \right\}.$$

We could also work as in a slightly different situation by considering the subfunctor $\mathfrak{X}_D^{\phi,0} := \mathfrak{X}_D^\phi \times_{\mathfrak{X}_D} \mathfrak{X}_D^0 \subset \mathfrak{X}_D^\phi$ parameterizing deformations with constant Hodge-Tate-Sen weight zero.

LEMMA 3.32 ([**43**, Proposition 8.13]). *Assume that* $\dim_L D_{\mathrm{cris}}^+(D)^{\varphi=\phi} = 1$*. Then,* $\mathfrak{X}_D^\phi \to \mathfrak{X}_D$ *is a relatively representable subfunctor of* $\mathfrak{X}_D$*.*

We hereafter denote by $\mathfrak{t}_D^\phi$ the Zariski tangent space to the functor $\mathfrak{X}_D^\phi$. The following lemma is clear but is going to be important for us.

LEMMA 3.33. *Suppose that* ϕ *is a simple eigenvalue for* φ *acting on* $D_{\mathrm{cris}}(D)$*. Assume as well that* $D = D_0 \oplus D_1$ *with* $D_{\mathrm{cris}}(D_0)^{\varphi=\phi} \neq (0)$*. Then, the natural projection* $\mathfrak{t}_D \to \mathfrak{t}_{D_0}$ *maps* $\mathfrak{t}_D^\phi$ *into* $\mathfrak{t}_{D_0}^\phi$*.*

PROOF. Let $\widetilde{D} \in \mathfrak{t}_D^\phi$ and choose $\widetilde{\phi}$ such that $D_{\mathrm{cris}}^+(\widetilde{D})^{\varphi=\widetilde{\phi}}$ is free of rank one over A. We use an explicit description of the natural map

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) = \mathfrak{t}_D \to \mathfrak{t}_{D_0} = \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_0, D_0)$$

in terms of extension classes. To be precise, the element of $\mathfrak{t}_{D_0}$ which $\widetilde{D}$ maps to is $\widetilde{D}_0 := \ker(\widetilde{D} \to D_1)/D_1$.

Since $\widetilde{D} \in \mathfrak{t}_D^\phi$, we know that there exists an embedding $\mathcal{R}_{L[\varepsilon]}(\mathrm{unr}(\widetilde{\phi})) \hookrightarrow \widetilde{D}$. By the multiplicity one of ϕ in $D_{\mathrm{cris}}(D)$, it induces an embedding $\mathcal{R}_{L[\varepsilon]}(\mathrm{unr}(\widetilde{\phi})) \hookrightarrow \widetilde{D}_0$. By the left exactness of D_{cris} we have

$$D_{\mathrm{cris}}(\mathcal{R}_{L[\varepsilon]}(\mathrm{unr}(\widetilde{\phi}))) \subset D_{\mathrm{cris}}(\widetilde{D}_0)^{\varphi=\widetilde{\phi}}. \tag{3.10}$$

On the other hand, $\dim_L D_{\mathrm{cris}}(\widetilde{D}_0)^{\varphi=\widetilde{\phi}} \leq 2$ because, again, ϕ is a simple eigenvalue. Thus (3.10) is an equality and $D_{\mathrm{cris}}(\widetilde{D}_0)^{\varphi=\widetilde{\phi}}$ is free of rank one. $\square$

REMARK. Under the notation above, we have as well that the projection $\mathfrak{t}_D \to \mathfrak{t}_{D_0}$ maps $\mathfrak{t}_D^{\phi,0}$ into $\mathfrak{t}_{D_0}^{\phi}$. However, note that 0 need not be a Hodge-Tate weight of D_0 and so one must be careful not to include that condition in the notation.

We now switch our focus to a more general setting. Let D be a (φ,Γ)-module over $\mathcal{R}_L$ and we assume that it is crystalline with regular Hodge-Tate weights $k_1 < k_2 < \ldots < k_n$ (though we no longer assume that $k_1 = 0$). If D_A is a deformation of D to A then we denote by $\kappa_1,\ldots,\kappa_n \in A$ the Hodge-Tate-Sen weights of D_A, labeled so that $\kappa_i \equiv k_i \bmod \mathfrak{m}_A$. Since $\mathfrak{m}_A$ is nilpotent there is a unique character $\mathbf{Q}_p^\times \to A^\times$ which we denote by as well by κ_i whose weight is $-\kappa_i$ and which is trivial at p. If D_A is a deformation then $D_A(\kappa_1)$ is a deformation of $D(k_1)$ and which has zero as a Hodge-Tate weight (this notation was first introduced in Example 2.4).

Recall we defined refinements in §2.3. We fix a refinement R of D corresponding to an ordering $(\phi_1,\ldots,\phi_n)$ of crystalline eigenvalues. Since D has distinct crystalline eigenvalues, we know that such an ordering corresponds exactly to a triangulation of D. Define $F_i := p^{-k_i}\phi_i \in L^\times$. Then, for each $i = 1,\ldots,n$ we have that the product $F_1\cdots F_i$ is a crystalline eigenvalue for φ acting on $\wedge^i D(k_1+\cdots+k_i)$. This is a (φ,Γ)-module with lowest Hodge-Tate weight 0 and so we can apply our previous discussion.

DEFINITION. *We say that a deformation D_A of D to A is of Kisin-type (with respect to R) if there exists elements $\Phi_1,\ldots,\Phi_n \in A^\times$ such that $\Phi_i \equiv F_i \bmod \mathfrak{m}_A$ for all $i = 1,\ldots,n$ and one has*

$$D^+_{\mathrm{cris}}(\wedge^i D_A(\kappa_1+\cdots+\kappa_i))^{\varphi=\Phi_1\cdots\Phi_i}$$

is free of rank one over A. We let $\mathfrak{X}^h_{D,R}$ denote the formal deformation functor of Kisin type.

The h is in homage to Kisin's original functor. In fact, the above is the natural generalization we mentioned at the beginning of this subsection. We have the following positive result regarding the representability of the above functor.

PROPOSITION 3.34. *Assume that for $i = 1,\ldots,n$ the eigenvalue $\phi_1\cdots\phi_i$ is multplicity one on $D_{\mathrm{cris}}(\wedge^i D)$. Then $\mathfrak{X}^h_{D,R}$ is a relatively representable subfunctor of $\mathfrak{X}_D$ and*

$$\mathfrak{t}^h_{D,R} = \ker\left(\mathfrak{t}_D \to \bigoplus_{i=1}^n \mathfrak{t}^0_{\wedge^i D(k_1+\cdots+k_i)}/\mathfrak{t}^{F_1\cdots F_i,0}_{\wedge^i D(k_1+\cdots+k_i)}\right). \tag{3.11}$$

PROOF. Under the assumption that $\phi_1\cdots\phi_i$ is a simple eigenvalue on $D^+_{\mathrm{cris}}(\wedge^i D(k_1+\cdots+k_i))$, we know by Lemma 3.32 that each containment

$$\mathfrak{X}^{F_1\cdots F_i,0}_{\wedge^i D(k_1+\cdots+k_i)} \subset \mathfrak{X}^0_{\wedge^i D(k_1+\cdots+k_i)}$$

defines a relatively representable subfunctor. Then, $\mathfrak{X}^h_{D,R}$ is defined as a fibered product

$$\begin{array}{ccc} \mathfrak{X}^h_{D,R} & \longrightarrow & \mathfrak{X}_D \\ \downarrow & & \downarrow \\ \prod_{i=1}^n \mathfrak{X}^{F_1\cdots F_i,0}_{\wedge^i D(k_1+\cdots+k_i)} & \longrightarrow & \prod_{i=1}^n \mathfrak{X}^0_{\wedge^i D(k_1+\cdots+k_i)} \end{array}$$

where the right hand vertical map is the map

$$D_A \mapsto \left(\wedge^i D_A(\kappa_{A,1}+\cdots+\kappa_{A,i})\right)_{i=1}^n.$$

Since the bottom horizontal arrow is relatively representable, so is the top arrow. The calculation of the tangent space is clear from this as well. ☐

We also have an inclusion $\mathfrak{X}_{D,f} \subset \mathfrak{X}_{D,R}^h$. Indeed, crystalline deformations have integer Hodge-Tate weights and so the twisting in the definition of $\mathfrak{X}_{D,R}^h$ can always be undone. Note that this containment holds for any choice of refinement, not just the non-critical ones (cf. Proposition 3.29).

REMARK. The arrow appearing in the definition

$$\mathfrak{t}_{D,R}^h = \ker\left(\mathfrak{t}_D \to \bigoplus_{i=1}^n \mathfrak{t}^0_{\wedge^i D(k_1+\cdots+k_i)}/\mathfrak{t}^{F_1\cdots F_i,0}_{\wedge^i D(k_1+\cdots+k_i)}\right)$$

is far from surjective, unless $n = 2$. The following lemma also shows that the appearance of the top exterior power in (3.11) is superfluous.

LEMMA 3.35. *Let $\delta : \mathbf{Q}_p^\times \to L^\times$ be a crystalline character of weight k and crystalline eigenvalue $\phi \in L^\times$. Then, $\mathfrak{t}^k_{\mathcal{R}_L(\delta)} = \mathfrak{t}^{\phi,k}_{\mathcal{R}_L(\delta)} = \mathfrak{t}_{\mathcal{R}_L(\delta),f} = \mathfrak{t}^{\phi}_{\mathcal{R}_L(\delta)}$.*

PROOF. Without loss of generality $\delta = 1$. Then, we have to show that if $\widetilde{\delta} : \mathbf{Q}_p^\times \to L[\varepsilon]^\times$ is a deformation of the trivial character with constant weight zero, it is crystalline. However, if $\widetilde{\delta}$ has weight zero then $\eta := \widetilde{\delta}\big|_{\mathbf{Z}_p^\times}$ is finite order and $\widetilde{\delta}$ is crystalline if and only if $\eta = 1$. We now show $\eta = 1$. For, if $z \in \mathbf{Z}_p^\times$ then we write $\eta(z) = 1 + a(z)\varepsilon$. This defines a group homomorphism $a : \mathbf{Z}_p^\times \to L$ (the additive group of L). Moreover, η has finite order if and only if a has finite order. Since L is divisible, this is not possible unless $a = 0$. ☐

The corollary we previously alluded to is the following.

COROLLARY 3.36. $\mathfrak{t}_{D,R}^h = \ker\left(\mathfrak{t}_D \to \bigoplus_{i=1}^{n-1} \mathfrak{t}^0_{\wedge^i D(k_1+\cdots+k_i)}/\mathfrak{t}^{F_1\cdots F_i,0}_{\wedge^i D(k_1+\cdots+k_i)}\right)$.

3.2.4. Putting it all together. To end this section on deformation theory we prove a theorem on the dimension of a certain deformation ring which we expect to be very close to the deformations appearing in a p-adic family. It will pull together all the deformation problems we have considered: crystalline deformations, paraboline deformations and deformations of Kisin-type.

Fix first a regular, crystalline (φ,Γ)-module D over $\mathcal{R}_L$ of rank n, say. Then, we make a choice $P = (P_i)_{i=1}^n$ of a triangulation for D. Denote by $(\delta_1,\ldots,\delta_n)$ the parameter associated to P and by $(\phi_1,\ldots,\phi_n)$ its associated ordering of Frobenius eigenvalues on φ. As in §2.3.1 we have its maximal non-critical parabolization P^{nc} associated to P. Denote by s the length of P^{nc}. Thus we have the deformation functor $\mathfrak{X}_{D,P^{\mathrm{nc}}}$ associated to this parabolization from §3.2.2. Moreover, we consider the map

$$\mathfrak{X}_{D,P^{\mathrm{nc}}} \to \prod_{i=1}^s \mathfrak{X}_{\mathrm{Gr}_j P^{\mathrm{nc}}}. \tag{3.12}$$

we first encountered in (3.9). We furthermore make the following regularity hypotheses on the crystalline eigenvalues:

(a) $\phi_i\phi_j^{-1} \notin \{1,p\}$ for any $i \neq j$.
(b) The eigenvalue $\phi_1\cdots\phi_i$ is multiplicty one on $D_{\mathrm{cris}}(\wedge^i D)$ for $i = 1,\ldots,n$.

Notice that while the former hypothesis is independent of P, the latter is not.

What we do now is use the paraboline deformations in concert with the Kisin-type deformations to make a successive cutting down of our deformation problem. That is, we define a deformation problem $\mathfrak{X}_{D,P}^{\mathrm{par},\wedge}$ (notice that we decorate it *with the triangulation* for emphasis) by the following fibered product

$$\begin{array}{ccc} \mathfrak{X}_{D,P}^{\mathrm{par},\wedge} & \longrightarrow & \mathfrak{X}_{D,P^{\mathrm{nc}}} \\ \downarrow & & \downarrow \\ \prod_{i=1}^{s} \mathfrak{X}^{h}_{\mathrm{Gr}_j P^{\mathrm{nc}}, R_j} & \longrightarrow & \prod_{i=1}^{s} \mathfrak{X}_{\mathrm{Gr}_j P^{\mathrm{nc}}} \end{array} \tag{3.13}$$

Here, R_j is the refinement of $\mathrm{Gr}_j P^{\mathrm{nc}}$ induced from the triangulation P of D.

LEMMA 3.37. *$\mathfrak{X}_{D,P}^{\mathrm{par},\wedge} \subset \mathfrak{X}_D$ is relatively representable.*

PROOF. Hypothesis (b) and Proposition 3.34 together imply that the bottom arrow defines a relatively representable subfunctor. Thus the same is true for the top arrow. Finally hypothesis (a) and Corollary 3.26 also imply that $\mathfrak{X}_{D,P^{\mathrm{nc}}}$ is a relatively representable subfunctor of $\mathfrak{X}_D$ and thus we are done. $\square$

Since $\mathfrak{X}_{D,P}^{\mathrm{par},\wedge}$ is relatively representable, it has a well-defined tangent space. By definition, we get a short exact sequence

$$0 \to \mathfrak{t}_{D,P}^{\mathrm{par},\wedge} \to \mathfrak{t}_{D,P^{\mathrm{nc}}} \to \bigoplus_{i=1}^{s} \mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}}}/\mathfrak{t}^{h}_{\mathrm{Gr}_j P^{\mathrm{nc}},R_j} \to 0. \tag{3.14}$$

The final arrow is surjective because hypothesis (a) implies that the right vertical arrow of (3.13) is surjective on points. The middle term is computed by Proposition 3.27. The quotient is what we were discussing in the remark proceeding Proposition 3.34. Thus, by inductively computing the quotients appearing in the direct sum, we can manage to compute the dimension of $\mathfrak{t}_{D,P}^{\mathrm{par},\wedge}$. Since P^{nc} is non-critical, Proposition 3.29 implies that $\mathfrak{t}_{D,f} \subset \mathfrak{t}_{D,P^{\mathrm{nc}}}$.

We now finish with the promised theorem. If E is a crystalline (φ,Γ)-module and R is a refinement of E then we will consider the following hypothesis:

$$\mathfrak{X}^{h}_{E,R} \text{ is a relatively representable subfunctor of } \mathfrak{X}_E \text{ and } \dim \mathfrak{t}^{h}_{E,R}/\mathfrak{t}_{E,f} \leq \mathrm{rank}_{\mathcal{R}_L} E. \tag{3.15}$$

THEOREM 3.38. *Assume D has regular weights and satisfies (a)-(c). Assume as well that for all j, $\mathrm{Gr}_j P^{\mathrm{nc}}$ together with its induced refinement R_j satisfy the hypothesis* (3.15)*. Then, we have that*

$$\dim \mathfrak{t}_{D,P}^{\mathrm{par},\wedge}/\mathfrak{t}_{D,f} \leq \mathrm{rank}_{\mathcal{R}_L} D.$$

Further, (3.15) *is an equality for all $\mathrm{Gr}_j P^{\mathrm{nc}}$ if and only if $\dim \mathfrak{t}_{D,P}^{\mathrm{par},\wedge}/\mathfrak{t}_{D,f} = \mathrm{rank}_{\mathcal{R}_L} D$.*

Recall that Lemma 2.26 implies that each $\mathrm{Gr}_j P^{\mathrm{nc}}$ is fully critical. Thus, the assumption of the theorem is about the deformation theory of fully critical refinements of crystalline (φ,Γ)-modules. Instance where this hypothesis is satisfied are given in §3.3.

PROOF OF THEOREM. To fix notation we let $n_j := \mathrm{rank}_{\mathcal{R}_L} \mathrm{Gr}_j P^{\mathrm{nc}}$. Then we know that

$$\dim \mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}}} = n_j^2 + \dim H^0(\mathrm{ad}\,\mathrm{Gr}_j P^{\mathrm{nc}})$$

by the Euler-Poincaré-Tate characteristic formula (Proposition 2.16) and Proposition 3.28 (please apply it with the trivial parabolization of $\mathrm{Gr}_j P^{\mathrm{nc}}$). On the other hand,

$$\dim \mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}},f} = \dim H^0(\mathrm{ad}\,\mathrm{Gr}_j P^{\mathrm{nc}}) + \binom{n_j}{2}$$

by Proposition 3.5 and Proposition 3.28. Thus the assumption (3.15) implies that

$$\begin{aligned}\dim \mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}}}/\mathfrak{t}^h_{\mathrm{Gr}_j P^{\mathrm{nc}},R_j} &= \dim \mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}}}/\mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}},f} - \dim \mathfrak{t}^h_{\mathrm{Gr}_j P^{\mathrm{nc}},R_j}/\mathfrak{t}_{\mathrm{Gr}_j P^{\mathrm{nc}},f} \\ &\geq n_j^2 + \binom{n_j}{2} - n_j.\end{aligned}$$

Turning to D itself, Proposition 3.27 and Proposition 3.28 (again) imply that

$$\dim \mathfrak{t}_{D,P^{\mathrm{nc}}} = \dim \mathrm{End}_{(\varphi,\Gamma)}(D) + \sum_{i\leq j} n_i n_j.$$

Let $d := \dim \mathrm{End}_{(\varphi,\Gamma)}(D)$. Plugging all of this into (3.14), we get

$$\begin{aligned}\dim \mathfrak{t}^{\mathrm{par},\wedge}_{D,P} &\leq d + \sum_{i\leq j} n_i n_j - \sum_{i=1}^{s}\left(n_i^2 + \binom{n_i}{2} - n_i\right) \\ &= d + \sum_{i=1}^{s}\left(n_i + \binom{n_i}{2} + \sum_{i<j} n_i n_j\right) \\ &= d + n + \sum_{i=1}^{s}\left(\binom{n_i}{2} + \sum_{i<j} n_i n_j\right),\end{aligned}$$

and hence

$$\begin{aligned}\dim \mathfrak{t}^{\mathrm{par},\wedge}_{D,P}/\mathfrak{t}_{D,f} &\leq d + n + \sum_{i=1}^{s}\left(\binom{n_i}{2} + \sum_{i<j} n_i n_j\right) - \left(d + \binom{n}{2}\right) \\ &= n + \sum_{i=1}^{s}\left(\binom{n_i}{2} + \sum_{i<j} n_i n_j\right) - \binom{n}{2}.\end{aligned}$$

The question is then settled by the fullowing numerical identity. □

LEMMA 3.39. *Let* $n = \sum_{i=1}^{s} n_i$. *Then,*

$$\binom{n}{2} = \sum_{i=1}^{s}\binom{n_i}{2} + \sum_{i<j} n_j n_j.$$

PROOF. We prove it by induction on s, the case of $s = 1$ being clear. If $s > 1$ we let $n' := n - n_1 = \sum_{i=2}^{s} n_i$. By induction then

$$\begin{aligned}\sum_{i=1}^{s}\binom{n_i}{2} + \sum_{i<j} n_i n_j &= \binom{n_1}{2} + n_1\sum_{i=2}^{s} n_j + \binom{n-n_1}{2} \\ &= \frac{1}{2}\left(n_1(n_1-1) + 2n_1(n-n_1) + (n-n_1)(n-n_1-1)\right) \\ &= \frac{1}{2}\left(n^2 - n\right).\end{aligned}$$

□

This completes our main explanation of the deformation theory of (φ,Γ)-modules. In the next section we present some cases where the hypothesis of this final theorem have been checked.

3.3. Computations of Kisin-type deformation functors for fully critical refinements

In this short section we collect some explicit calculations of the tangent spaces to Kisin-type deformation functors. We show that the hypothesis of Theorem 3.38 is satisfied if (in the notation there) the associated gradeds $\mathrm{Gr}_j P^{\mathrm{nc}}$ have

(a) rank at most two, or
(b) rank three and if $\mathrm{Gr}_j P^{\mathrm{nc}}$ is a sum of three characters then the associated weight ordering (s_1, s_2, s_3) is not strictly decreasing.

This list is far from exhaustive, so please consider it just a sampling of what is possible.

As an example then, we have the following corollary. It will be used to give new examples of smooth points on p-adic families (see Chapter 5).

Proposition 3.40. *Suppose D is an indecomposable rank three crystalline (φ,Γ)-module with regular weights and distinct crystalline eigenvalues. Suppose as well that for each pair of eigenvalues $\phi \neq \phi'$ we have $\phi \neq p\phi'$. Then, for any choice of triangulation P on D we have that* $\dim \mathfrak{t}_{D,P}^{\mathrm{par},\wedge}/\mathfrak{t}_{D,f} \leq 3$.

Proof. It suffices (by Theorem 3.38[4] and the case (a) above) to show that for any choice of triangulation P, P^{nc} is non-trivial. Let $k_1 < k_2 < k_3$ be the Hodge-Tate weights of D and choose a triangulation

$$P : 0 \subsetneq P_1 \subsetneq P_2 \subsetneq D.$$

Denote by $(\delta_1, \delta_2, \delta_3)$ the associated parameter. Suppose that both P_1 and P_2 are critical and we will show that D is decomposable. By duality, we may assume without loss of generality that $\mathrm{wt}(\delta_1) = k_3$. Indeed, if P_1 is critical then either $\mathrm{wt}(\delta_1) = k_2$ or $\mathrm{wt}(\delta_1) = k_3$. If we are in the first case, then since P_2 is critical we must have $\mathrm{wt}(\delta_2) = k_3$ and $\mathrm{wt}(\delta_3) = k_1$. Thus $D^\vee$ has the dual parameter $(\delta_i^{-1})_{i=1}^3$ and $\mathrm{wt}(\delta_1^{-1})$ is the highest Hodge-Tate weight.

Let $Q_2 = D/\mathcal{R}_L(\delta_1)$. Then, by definition we may see D as an element of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(Q_2, \mathcal{R}_L(\delta_1)) \cong H^1(Q_2^\vee(\delta_1))$. Moreover, since D is crystalline it defines a class inside the Bloch-Kato Selmer group $H^1_f(Q_2^\vee(\delta_1)) \subset H^1(Q_2^\vee(\delta_1))$. However, $Q_2^\vee(\delta_1)$ has two Hodge-Tate weights $k_3 - k_2$ and $k_3 - k_1$, each of which is positive. Thus, Propostion 2.19 implies that

$$\dim H^1_f(Q_2^\vee(\delta_1)) = \dim H^0(Q_2^\vee(\delta_1)) = 0.$$

The last equality following from the regularity of the eigenvalues. Thus, D must be split. ☐

In the following sections we carefully compute $\mathfrak{t}^h_{D,R}$ with respect to fully critical refinements.

3.3.1. The rank one case. It is sort of silly (there are no critical refinements) to describe this, but here it goes anyways. Let δ be a crystalline character of weight $k \in \mathbf{Z}$. Then we have the following facts:

(1) $\mathrm{ad}\,\mathcal{R}_L(\delta) \cong \mathcal{R}_L$ as (φ,Γ)-modules.
(2) There is only one triangulation $P : 0 \subsetneq \mathcal{R}_L(\delta)$; it is non-critical.
(3) By Proposition 2.1, any deformation D_A to A is of the form $\mathcal{R}_A(\delta_A)$ and is thus a trianguline deformation.
(4) Equivalently, $D_{\mathrm{cris}}(\mathcal{R}_A(\delta_A)(\mathrm{wt}\,\delta_A))^{\varphi=\delta_A(p)}$ is always free of rank one.

In particular we have that $\mathfrak{t}^0_{\mathcal{R}_L(\delta)(\mathrm{wt}\,\delta)} = \mathfrak{t}^{\delta(p),0}_{\mathcal{R}_L(\delta)(\mathrm{wt}\,\delta)}$ and thus (3.14) says that

$$\mathfrak{t}^{\mathrm{par},\wedge}_{\mathcal{R}_L(\delta),P} = \ker\left(\mathfrak{t}_{\mathcal{R}_L(\delta)} \to 0\right) = \mathfrak{t}_{\mathcal{R}_L(\delta)}.$$

[4]Notice that the hypothesis on the second exterior power $\wedge^2 D$ can be seen as a hypothesis on $D^\vee$, up to a twist. It is easy to see that our assumptions are enough to conclude that condition (b) preceding Theorem 3.38 is satisfied.

In particular, $\mathfrak{t}^{\mathrm{par},\wedge}_{\mathcal{R}_L(\delta),P}/\mathfrak{t}_{\mathcal{R}_L(\delta),f} = \mathfrak{t}_{\mathcal{R}_L(\delta)}/\mathfrak{t}_{\mathcal{R}_L(\delta),f}$ is one-dimensional (by Proposition 2.19). We could have also used Corollary 3.36 to conclude the same thing.

3.3.2. Critical rank two triangulations. Let D be a rank two crystalline (φ,Γ)-module over $\mathcal{R}_L$ with regular Hodge-Tate weights $k_1 < k_2$ and crystalline eigenvalues $\phi_1 \neq \phi_2$. Assume that D has a critical refinement. By Example 2.22 this is equivalent to D being split.

Label the crystalline eigenvalues (ϕ_1,ϕ_2) so that it defines a critical refinement R_{crit} of D. We write the associated parameter is $(\delta_1,\delta_2) = (z^{-k_2}\operatorname{unr}(\phi_1), z^{-k_1}\operatorname{unr}(\phi_2))$ and we assume as well now that $\phi_1 \neq p^{\pm 1}\phi_2$.

We now begin to show that (3.15) is valid for D with respect to R_{crit}. Note that by Corollary 3.36 we have that

$$\mathfrak{t}^h_{D,R_{\mathrm{crit}}} = \ker\left(\mathfrak{t}_D \to \mathfrak{t}^0_{D(k_1)}/\mathfrak{t}^{\delta_1(p),0}_{D(k_1)}\right).$$

Since D is split, $\operatorname{ad} D$ is also split and we have a decomposition

$$\mathfrak{t}_D = H^1(\operatorname{ad} D) = \bigoplus_{i,j=1}^{2} H^1(\mathcal{R}_L(\delta_i\delta_j^{-1})).$$

In §2.1.2 we defined functors D^+_{dif} and D_{Sen} which fit into a diagram of categories:

$$\begin{array}{ccc} (\varphi,\Gamma)_{/R_L} & \xrightarrow{D^+_{\mathrm{dif}}} & \mathbf{Q}_p(\mu_{p^\infty})[[t]][\Gamma]\text{-mod} \\ \downarrow{\scriptstyle D_{\mathrm{Sen}}} & \swarrow{\scriptstyle \otimes/t} & \\ \mathbf{Q}_p(\mu_{p^\infty})[\Gamma]\text{-mod} & & \end{array}$$

On the level of cohomology groups, we have morphisms

$$\begin{array}{ccccc} H^1(\operatorname{ad} D) & \xrightarrow{D^+_{\mathrm{dif}}} & \operatorname{Ext}^1(D^+_{\mathrm{dif}}(D), D^+_{\mathrm{dif}}(D)) & \xrightarrow{\cong} & \bigoplus_{i,j=1}^2 H^1(\Gamma, D^+_{\mathrm{dif}}(\delta_i\delta_j^{-1})) \\ \| & & \downarrow & & \downarrow \\ H^1(\operatorname{ad} D) & \xrightarrow{D_{\mathrm{Sen}}} & \operatorname{Ext}^1(D_{\mathrm{Sen}}(D), D_{\mathrm{Sen}}(D)) & \xrightarrow{\cong} & \bigoplus_{i,j=1}^2 H^1(\Gamma, D_{\mathrm{Sen}}(\delta_i\delta_j^{-1})) \end{array} \tag{3.16}$$

The cohomology groups in the right column are easily computed. We have

$$\dim_{\mathbf{Q}_p} H^1(\Gamma, t^n\mathbf{Q}_p(\mu_{p^\infty})) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

By dévissage we get that

$$\dim_{\mathbf{Q}_p} H^1(\Gamma, t^n\mathbf{Q}_p(\mu_{p^\infty})[[t]]) = \begin{cases} 1 & \text{if } n \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

In particular, putting the coefficient structure back in we have that

$$H^1(\Gamma, D_{\mathrm{Sen}}(\delta_i\delta_i^{-1})) \cong L. \tag{3.19}$$

Suppose that $\widetilde{D} \in H^1(\operatorname{ad} D)$. Write $\widetilde{\kappa}_i = k_i + b_i\varepsilon$ for the Hodge-Tate-Sen weight deforming k_i. We recall from §3.2.2 that the constant weight deformations are given by

$$\mathfrak{t}^{k_i}_D = \ker\left(H^1(\operatorname{ad} D) \to H^1(\Gamma, D_{\mathrm{Sen}}(\delta_i\delta_i^{-1}))\right). \tag{3.20}$$

The key lemma for computing $\mathfrak{t}^{\phi_1}_D$ is the following

LEMMA 3.41. *If $\widetilde{D} \in \mathfrak{t}^h_{D,R_{\mathrm{crit}}}$. then $\widetilde{\kappa}_2 - \widetilde{\kappa}_1 = k_2 - k_1$ is constant.*

Proof. Write $\widetilde{D}'$ for $\widetilde{D}(\widetilde{\kappa}_1)$. Recall that $F_1 = p^{-k_1}\phi_1$ and we choose $\widetilde{\Phi} \in L[\varepsilon]^\times$ such that $\widetilde{\Phi} = F_1 + a\varepsilon$ and that $D_{\mathrm{cris}}(\widetilde{D}')^{\varphi=\widetilde{\Phi}}$ is free of rank one. Our goal is to show that $\widetilde{D}'$ (which is an element of $\mathfrak{t}^0_{D(k_1)}$) is actually inside the subspace $\mathfrak{t}^{k_2-k_1}_{D(k_1)}$.

Consider the factorization (in the notation of §3.2.2)

$$\begin{array}{ccl} \mathfrak{t}_{D(k_1)} & \longrightarrow & \mathrm{Ext}^1\left(D_{\mathrm{Sen}}(D(k_1))_{(k_2-k_1)}, D_{\mathrm{Sen}}(D(k_1))_{(k_2-k_1)}\right) . \\ \downarrow & \nearrow & \\ \mathfrak{t}_{\mathcal{R}_L(\delta_1)(k_1)} & & \end{array}$$

Since $\mathfrak{t}^{k_2-k_1}_{D(k_1)}$ is the kernel of the top horizontal map, we will check its image under the projection $\mathfrak{t}_{D(k_1)} \to \mathfrak{t}_{\mathcal{R}_L(\delta_1)(k_1)}$ lands inside the constant weight subspace $\mathfrak{t}^{k_2-k_1}_{\mathcal{R}_L(\delta_1)(k_1)}$. To see this note that by Lemma 3.33 we know that the projector $\mathfrak{t}_{D(k_1)} \to \mathfrak{t}_{\mathcal{R}_L(\delta_1)(k_1)}$ induces a map on subspaces $\mathfrak{t}^{\delta_1(p),0}_{D(k_1)} \to \mathfrak{t}^{\delta_1(p)}_{\mathcal{R}_L(\delta_1)(k_1)}$. We then apply Lemma 3.35 to see that $\mathfrak{t}^{\delta_1(p)}_{\mathcal{R}_L(\delta_1)(k_1)} = \mathfrak{t}^{k_2-k_1}_{\mathcal{R}_L(\delta_1)(k_1)}$. □

Following this result we can compute $\mathfrak{t}^h_{D,R_{\mathrm{crit}}}$.

Proposition 3.42. *There is a diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{t}_{D,f} & \longrightarrow & \mathfrak{t}^h_{D,R_{\mathrm{crit}}} & \longrightarrow & \mathrm{Ext}^1_\Gamma(D^+_{\mathrm{dif}}(D), D^+_{\mathrm{dif}}(D)) \\ & & \| & & \uparrow & & \| \\ 0 & \longrightarrow & \mathfrak{t}_{D,f} & \longrightarrow & \mathfrak{t}^{h,k_1}_{D,R_{\mathrm{crit}}} & \longrightarrow & \mathrm{Ext}^1_\Gamma(D^+_{\mathrm{dif}}(D), D^+_{\mathrm{dif}}(D)) \end{array}$$

with exact rows. The image of the final map is at most two-dimensional (resp. one-dimensional) in the top (resp. bottom) row.

Proof. First, the assumptions we have on the crystalline eigenvalues implies that we have that

$$\mathfrak{t}_{D,f} = \mathfrak{t}_{D,g} = \ker(\mathfrak{t}_D \to \mathrm{Ext}^1(D^+_{\mathrm{dif}}(D), D^+_{\mathrm{dif}}(D)).$$

Furthermore, $\mathfrak{t}_{D,f} \subset \mathfrak{t}^{h,k_1}_{D,R}$, hence the exact sequence. It remains for us to compute the image.

Let $*$ be either Sen or dif^+ and set

$$E_*(i,j) = \mathrm{Ext}^1(D_*(\mathcal{R}_L(\delta_i)), D_*(\mathcal{R}_L(\delta_j))).$$

We then consider a diagram

$$\begin{array}{ccccc} \mathfrak{t}^h_{D,R_{\mathrm{crit}}} & & & & \\ & \searrow & & & \\ & & \mathrm{Ext}^1(D^+_{\mathrm{dif}}(D), D^+_{\mathrm{dif}}(D)) & \xrightarrow{\cong} & E^+_{\mathrm{dif}}(1,1) \oplus E^+_{\mathrm{dif}}(2,2) \oplus E^+_{\mathrm{dif}}(1,2) \\ & & \downarrow & & \downarrow {\scriptstyle (\cong,\cong,0)} \\ & & \mathrm{Ext}^1(D_{\mathrm{Sen}}(D), D_{\mathrm{Sen}}(D)) & \xrightarrow[\cong]{} & E_{\mathrm{Sen}}(1,1) \oplus E_{\mathrm{Sen}}(2,2) \end{array}$$

The isomorphisms are justified by (3.17) and (3.18). Furthermore, the same computations imply that each direct summand in the right hand column is one-dimensional.

If $\widetilde{D} \in \mathfrak{t}^h_{D,R_{\mathrm{crit}}}$ then we just saw in Lemma 3.41 that the image of $\widetilde{D}$ in the bottom right column is contained in diagonal subspace $\{(a,a) \colon a \in L\}$ and hence its image in the upper right corner is contained in the two-dimensional space consisting of $E^+_{\mathrm{dif}}(1,2)$ and the diagonal subspace of

$E^+_{\mathrm{dif}}(1,1) \oplus E^+_{\mathrm{dif}}(2,2) = L^{\oplus 2}$. This shows the image in the top row has dimension at most two. In the case where $\widetilde{\kappa}_1 = k_1$ is a constant weight, we have that its image in the bottom right is zero and hence the image in the top right is contained in $E^+_{\mathrm{dif}}(1,2)$. □

This completes all the relevant computations in the rank two situation. We are now free to apply Theorem 3.38 to any (φ,Γ)-module with respect to a triangulation whose maximal non-critical parabolization has associated grades of rank at most two.

3.3.3. A three-dimensional, anti-ordinary case. Here we will present partial computations in the fully critical rank three case. We fix a rank three crystalline (φ,Γ)-module D over $\mathcal{R}_L$ with regular weights $k_1 < k_2 < k_3$, satisfying the hypotheses (a) and (b) of §3.2.4. Notice that the arguments in Proposition 3.40 show that if P is fully critical then D is necessarily split. We now make the following assumptions:

(i) D is decomposed into a direct sum $\mathcal{R}_L(\delta) \oplus D_0$ where $\mathrm{wt}(\delta) = k_3$.
(ii) P is a fully critical triangulation $0 \subsetneq \mathcal{R}_L(\delta) \subsetneq P_2 \subsetneq D$ such that the induced triangulation on $D/\mathcal{R}_L(\delta) \cong D_0$ is non-critical (notice that this is different than P_2 being non-critical!)
(iii) The hypothesis (a) and (b) from §3.2.4 holds.
(iv) We have $\mathrm{Hom}_{(\varphi,\Gamma)}(D_0(1), D_0) = (0)$ (this will be used in a technical manner, see Corollary 3.47).

Please note that if P orders the eigenvalues (ϕ_1,ϕ_2,ϕ_3) then the associated ordering of the weights must be $(s_1,s_2,s_3) = (k_3,k_1,k_2)$. Indeed, we know $s_1 = k_3$ because $\mathrm{wt}(\delta) = k_3$. If $s_2 = k_2$ then the induced triangulation on D_0 would be critical, which we have ruled out.

Remark. We know by definition that any fully critical triangulation P must order the weights (s_1,s_2,s_3) as one of

$$(k_3,k_1,k_2), \quad (k_2,k_3,k_1), \quad (k_3,k_2,k_1).$$

The first case is the one we are investigating here and the second is dual to the first. In the third case, D is necessarily split into a direct sum of characters and we have (thus far) been unable to verify (3.15) in that case. Note that we haven't ruled out that D is totally split in our hypothesis, just this ordering in the totally split case.

Recall that our goal is to prove the following theorem.

Theorem 3.43. $\dim_L \mathfrak{t}^h_{D,P}/\mathfrak{t}_{D,f} \leq 3$.

The proof is going to follow ideas very similar to that of §3.3.2. However, we will certainly have to make use of at least one higher exterior power. By Lemma 3.35 we can still ignore the top exterior power in the calculation of $\mathfrak{t}^h_{D,P}$.

Recall that the following well known fact. Suppose that S is a commutative ring and that M is a finite free module of rank n over S. Then there exists canonical isomorphisms $\wedge^p(M^\vee) \cong \wedge^p(M)^\vee$ for each $0 \leq p \leq n$. Moreover, this defines an isomorphism

$$\begin{aligned} \wedge^p(M^\vee) \otimes \wedge^n(M) &\to \wedge^{n-p}(M^\vee)^\vee \\ x \otimes y &\mapsto \left[x' \mapsto \langle x \wedge x', y\rangle\right] \end{aligned}$$

where $\langle , \rangle$ is the pairing $\det M \otimes \det(M^\vee) \to S$. In particular, there exists canonical isomorphisms

$$\wedge^p\left(M^\vee\right) \otimes_S \wedge^n M \xrightarrow{\cong} \wedge^{n-p} M.$$

The naturality implies that we have (φ,Γ)-equivariant isomorphisms

(a) $\wedge^2 D \cong D^\vee(\det D)$, and
(b) $D_0 \cong D_0^\vee(\det D_0)$.

Using this, we have the following direct sum decompositions

$$\begin{aligned} \mathfrak{t}_D &= \mathfrak{t}_{D_0} \oplus \mathfrak{t}_{\mathcal{R}_L(\delta)} \oplus H^1(D_0(\delta^{-1})) \oplus H^1(D_0^\vee(\delta)), \\ \mathfrak{t}_{D(k_1)} &= \mathfrak{t}_{D_0(k_1)} \oplus \mathfrak{t}_{\mathcal{R}_L(\delta)(k_1)} \oplus H^1(D_0(\delta^{-1})) \oplus H^1(D_0^\vee(\delta)), \text{and} \\ \mathfrak{t}_{\wedge^2 D(k_1+k_2)} &= \mathfrak{t}_{\det D_0(k_1+k_2)} \oplus \mathfrak{t}_{D_0(\delta)(k_1+k_2)} \oplus H^1(D_0(\delta(\det D_0)^{-1})) \oplus H^1(D_0^\vee(\delta^{-1}\det D_0)). \end{aligned}$$

Notice that we have kept the weight twisting in some components (e.g. $\mathfrak{t}_{D_0}(k_1)$ in the second line) but not in others (e.g. $H^1(D_0(\delta^{-1}))$ in the same line). This is just a matter of taste.

Recall that $F_i = p^{-k_i}\phi_i$. By Lemma 3.33 and Lemma 3.35 we have have that the projector $\mathfrak{t}_{D(k_1)} \to \mathfrak{t}_{\mathcal{R}_L(\delta)(k_1)}$ maps $\mathfrak{t}_{D(k_1)}^{F_1,0}$ into the crystalline subspace $\mathfrak{t}_{\mathcal{R}_L(\delta)(k_1),f}$. Furthermore, $D_0(\delta^{-1})$ has only negative Hodge-Tate weights $k_1 - k_3$ and $k_2 - k_3$. Thus, by Proposition 2.19 we have that $H^1_f(D_0(\delta^{-1})) = H^1(D_0(\delta^{-1}))$. On the other hand, $H^1_f(D_0^\vee(\delta)) = (0)$. It follows from these comments that one has an inclusion

$$\mathfrak{t}_{D(k_1)}^{F_1,0}/\mathfrak{t}_{D(k_1),f} \hookrightarrow \mathfrak{t}_{D_0(k_1)}^0/\mathfrak{t}_{D_0(k_1),f} \oplus H^1(D_0^\vee(\delta)). \tag{3.21}$$

Similar comments reveal that we also have an inclusion (note that $D_0(\delta(\det D_0)^{-1})$ has two positive Hodge-Tate weights $k_3 - k_1$ and $k_3 - k_2$)

$$\mathfrak{t}_{\wedge^2 D(k_1+k_2)}^{F_1F_2,0}/\mathfrak{t}_{\wedge^2 D(k_1+k_2),f} \hookrightarrow \mathfrak{t}_{D_0(\delta)(k_1+k_2)}^{F_1F_2}/\mathfrak{t}_{D_0(\delta)(k_1+k_2),f} \oplus H^1(D_0(\delta(\det D_0)^{-1})). \tag{3.22}$$

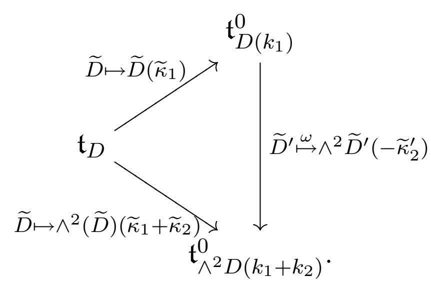
The next point is that we have as well as a diagonal map

$$\mathfrak{t}_D^h/\mathfrak{t}_{D,f} \to \mathfrak{t}_{D(k_1)}^{F_1,0}/\mathfrak{t}_{D(k_1),f} \oplus \mathfrak{t}_{\wedge^2 D(k_1+k_2)}^{F_1F_2,0}/\mathfrak{t}_{\wedge^2 D(k_1+k_2),f}. \tag{3.23}$$

We then will compute $\mathfrak{t}_D^h/\mathfrak{t}_{D,f}$ by

(a) understanding the kernel of (3.23) (see Corollary 3.46)
(b) understanding the image of (3.23) after postcomposing with (3.21) and (3.22).

The technique we are going to use for this is that there is a map $\omega : \mathfrak{t}_{D(k_1)}^0 \to \mathfrak{t}_{\wedge^2 D(k_1+k_2)}^0$ that fits into a diagram



Here, the notation $\widetilde{\kappa}_2'$ means the Hodge-Tate-Sen weight of $\widetilde{D}'$ which deforms the "second weight" $k_2 - k_1$ of $D(k_1)$ (the "first weight" is zero). It is easy to see that this diagram commutes. Thus, we can study the image of (3.23) by studying the image in the first direct summand then studying the image of that under ω. We now begin making our comptutations.

Lemma 3.44. *The map* $\omega : \mathfrak{t}_{D(k_1)}^0 \to \mathfrak{t}_{\wedge^2 D(k_1+k_2)}^0$ *induces an isomorphism*

$$H^1(D_0^\vee(\delta)) \xrightarrow{\cong} H^1(D_0(\delta^{-1}\det D_0)). \tag{3.24}$$

Proof. If we unwind the definition, we get that ω sends an element $\widetilde{E} \in H^1(D_0^\vee(\delta))$ arising from $\widetilde{D}' \in \mathfrak{t}_{D(k_1)}$ to the (φ,Γ)-module $\widetilde{F} := \widetilde{E}^\vee(\det \widetilde{D}')(\widetilde{\kappa}_2')$ inside the space $H^1(D_0(\delta^{-1}\det D_0))$. Since the dimension of each side of (3.24) is the same (by hypothesis (iii)) , it suffices to show that the map is injective. So, suppose that $\widetilde{F}$ is split. Since the weights of D_0 are different than the weight of δ we have that both $\widetilde{E}$ and $\widetilde{F}$ are Hodge-Tate. In particular, the character $\widetilde{\eta} = (\det \widetilde{D}')(\widetilde{\kappa}_2')$

is Hodge-Tate. But, since its reduction $\eta = (\det D)(k_1+k_2)$ is crystalline, η must also be crystalline by Lemma 3.35. Thus if $\widetilde{F}$ is split, so is $\widetilde{E}^\vee$. Finally, $\widetilde{E}$ must also be split then. □

The key result for us is going to be the following proposition. It provides us with the analog of Lemma 3.41.

PROPOSITION 3.45. *Suppose that* $\widetilde{D} \in \mathfrak{t}^h_{D,P}$ *with Hodge-Tate-Sen weights* $\widetilde{\kappa}_1, \widetilde{\kappa}_2$ *and* $\widetilde{\kappa}_3$*. Then, for all pairs* (i,j) *we have that* $\widetilde{\kappa}_i - \widetilde{\kappa}_j$ *is constant.*

PROOF. Consider the projection $\mathfrak{t}_{D(k_1)} \twoheadrightarrow \mathfrak{t}_{\mathcal{R}_L(\delta)(k_1)}$. By Lemma 3.33, the projection $\mathfrak{t}_{D(k_1)} \to \mathfrak{t}_{\mathcal{R}_L(\delta)(k_1)}$ maps $\mathfrak{t}^h_{D(k_1),P}$ into the subspace $\mathfrak{t}^{F_1}_{\mathcal{R}_L(\delta)(k_1)}$. The constancy of $\widetilde{\kappa}_3 - \widetilde{\kappa}_1$ then follows from Lemma 3.35.

It now suffices to show that $\widetilde{\kappa}_3 - \widetilde{\kappa}_2$ is constant. For this we consider the deformation condition on the second exterior power. Notice that $\wedge^2 D(k_1+k_2)$ contains $D_0(\delta)(k_1+k_2)$ as a (φ,Γ)-module direct summand and we consider the composition $\mathfrak{t}_D \to \mathfrak{t}_{\wedge^2 D(k_1+k_2)} \to \mathfrak{t}_{D_0(\delta)(k_1+k_2)}$. The image in the final subspace is in charge of (among other things) keeping track of the deformations $\widetilde{\kappa}_3 - \widetilde{\kappa}_2$ and $\widetilde{\kappa}_3 - \widetilde{\kappa}_1$ of the two weights $k_3 - k_2$ and $k_3 - k_1$ appearing in $\wedge^2 D(k_1+k_2)$. But notice now that $k_3 - k_2$ is the least Hodge-Tate weight of $D_0(\delta)(k_1+k_2)$ and, moreover, if $\widetilde{D} \in \mathfrak{t}^h_{D,P}$ then its image lands inside $\mathfrak{t}^{F_1F_2}_{D_0(\delta)(k_1+k_2)}$ by Lemma 3.33. By the constant weight lemma [**5**, Proposition 2.5.4] (see also the remarks proceeding Lemma 3.41), we have that $\widetilde{\kappa}_3 - \widetilde{\kappa}_2$ is constant. □

COROLLARY 3.46. *We have that*

$$\ker\left(\mathfrak{t}^h_{D,P} \to \mathfrak{t}^0_{D(k_1)} \oplus \mathfrak{t}^0_{\wedge^2 D(k_1+k_2)}\right) = \ker\left(\mathfrak{t}^h_{D,P} \to \mathfrak{t}^0_{D(k_1)}\right)$$

PROOF. Unwinding the equality we need to show that if $\widetilde{D} \in \mathfrak{t}^h_{D,P}$ such that $\widetilde{D}(\widetilde{\kappa}_1)$ is a constant deformation then $\wedge^2\widetilde{D}(\widetilde{\kappa}_1+\widetilde{\kappa}_2)$ is also constant. However, we have that

$$\wedge^2\widetilde{D}(\widetilde{\kappa}_1+\widetilde{\kappa}_2) = \wedge^2\left[\widetilde{D}(\widetilde{\kappa}_1)\right](\widetilde{\kappa}_2-\widetilde{\kappa}_1)$$

By Proposition 3.45 the final twist is a twist by an integer and thus the claim is clear. □

COROLLARY 3.47. *The composition* $\mathfrak{t}^h_{D,P} \to \mathfrak{t}^0_{\wedge^2 D(k_1+k_2)} \to \mathfrak{t}_{D_0(\delta)(k_1+k_2)}$ *has image inside the crystalline subspace* $\mathfrak{t}_{D_0(\delta)(k_1+k_2),f}$*.*

PROOF. Let $\widetilde{D} \in \mathfrak{t}^h_{D,P}$ and for the moment let us use the notation $\widetilde{D}'$ for its image in $\mathfrak{t}_{D_0(\delta)(k_1+k_2)}$. By Lemma 3.33 we know that $\widetilde{D}'$ must land inside $\mathfrak{t}^{F_1F_2}_{D_0(\delta)(k_1+k_2)}$. However, the refinement (F_1F_2, F_1F_3) of $D_0(\delta)(k_1+k_2)$ is non-critical by the assumption (ii) at the beginning of this subsection. Thus, $\widetilde{D}'$ is a trianguline deformation of $D_0(\delta)(k_1+k_2)$ with respect to this refinement. Furthermore, by Proposition 3.45 we know that $\widetilde{D}'$ is also Hodge-Tate. Thus we can apply [**5**, Theorem 2.5.1] to $\widetilde{D}'$—the hypothesis there being valid because of our hypothesis (iv) at the beginning of this subsection. The conclusion, is that $\widetilde{D}'$ is crystalline, which is what we wanted to show. □

COROLLARY 3.48. *The image of the composition* $\mathfrak{t}^h_{D,P} \to \mathfrak{t}^0_{D(k_1)} \to \mathfrak{t}^0_{D_0(k_1)}$ *lands inside* $\mathfrak{t}_{D_0(k_1),f}$*.*

Proof. We consider the following diagram whose dashed arrows we have explained

$$\begin{array}{ccccc} & & \mathfrak{t}_D & & \\ & \swarrow^{\text{twist}} & & \searrow^{\wedge^2(-)(\text{twist})} & \\ \mathfrak{t}^0_{D(k_1)} & & \overset{\omega}{\dashrightarrow} & & \mathfrak{t}^0_{\wedge^2 D(k_1+k_2)} \\ \downarrow\!\!\downarrow & & & & \downarrow\!\!\downarrow \\ \mathfrak{t}^0_{D_0(k_1)} & & \dashrightarrow & & \mathfrak{t}_{D_0(\delta)(k_1+k_2)}. \end{array}$$

The vertical arrows are projections. It is easy to see that the diagram commutes.

Now let $\widetilde{D} \in \mathfrak{t}^h_{D,P}$ and denote by $\widetilde{D}'$ its image in $\mathfrak{t}^0_{D(k_1)}$. We call $\widetilde{E}$ its image under the left vertical map (which we want to show is crystalline) and $\widetilde{F}$ its image in the bottom right corner. A diagram chase reveals that the relationship between these three is that

$$\widetilde{F} = \left(\widetilde{E}^\vee \otimes \det \widetilde{D}'\right)(\widetilde{\kappa}'_2).$$

By Corollary 3.47 we know that $\widetilde{F}$ is crystalline. We now as well that $\widetilde{\kappa}'_2 = \widetilde{\kappa}_2 - \widetilde{\kappa}_1$ is constant by Proposition 3.45. Thus, $\widetilde{E}^\vee(\det \widetilde{D}')$ is crystalline as well. However, $\widetilde{E}$ is rank two over $\mathcal{R}_{L[\varepsilon]}$ and thus we have that

$$\widetilde{E}^\vee(\det \widetilde{D}') = \widetilde{E}(\det \widetilde{D}'(\det \widetilde{E})^{-1})$$

Thus it suffices to show that $\det \widetilde{D}'(\det \widetilde{E})^{-1}$ is crystalline. However, this is a deformation of a crystalline character and its Hodge-Tate weight is $\widetilde{\kappa}'_3 = \widetilde{\kappa}_3 - \widetilde{\kappa}_1$. By Proposition 3.45 this is an integer and we are done. $\square$

We now put it all together to prove Theorem 3.43.

Proof of Theorem 3.43. First, notice that $\ker\left(\mathfrak{t}_D \to \mathfrak{t}^0_{D(k_1)}\right) \cap \mathfrak{t}_{D,f} = (0)$. In fact, if $\widetilde{D}$ is crystalline then $\widetilde{\kappa}_1 = k_1$. In that case, $\widetilde{D}(\widetilde{\kappa}_1)$ is split if and only if $\widetilde{D}$ is. In particular, we have an inclusion $\ker\left(\mathfrak{t}^h_D \to \mathfrak{t}^0_{D(k_1)}\right) \subset \mathfrak{t}^h_D/\mathfrak{t}_{D,f}$.

On the other hand, the three previous corollaries imply that we can compute a superspace for the cokernel; we express this as the short exact sequence

$$0 \to \ker\left(\mathfrak{t}^h_D \to \mathfrak{t}^0_{D(k_1)}\right) \to \mathfrak{t}^h_D/\mathfrak{t}_{D,f} \to H^1(D_0(\delta^{-1})).$$

In particular we have that

$$\dim \mathfrak{t}^h_D/\dim \mathfrak{t}_{D,f} \leq \dim H^1(D_0(\delta^{-1})) + \dim\left[\ker(\mathfrak{t}^h_D \to \mathfrak{t}^0_{D(k_1)})\right] \leq 3.$$

The final inequality follows because we clearly have that $\mathfrak{t}_D \twoheadrightarrow \mathfrak{t}^0_{D(k_1)}$ and thus

$$\ker(\mathfrak{t}^h_D \to \mathfrak{t}^0_{D(k_1)}) \subset \ker(\mathfrak{t}_D \to \mathfrak{t}^0_{D(k_1)}) \cong L,$$

the last isomorphism being a simple calculation of dimensions. $\square$

CHAPTER 4

Families of (φ, Γ)-modules

Fix an affinoid space $X = \mathrm{Sp}(A)$. We assume throughout that X is reduced. Recall that we have the notion of a (φ, Γ)-module over the space X from Chapter 1. In the present chapter, we will focus on examples which we call *refined families of* (φ, Γ)*-modules*. Such families arise naturally from the construction of p-adic families of automorphic forms. At each point in $x \in X$, the (φ, Γ)-module D_x will carry a canonical triangulation P_x and our main theorem will concern the analytic variation of these triangulations near *classical points*.

Previously, it has been understood [**43, 5**] that near classical, *non-critical* points x (points where P_x is non-critical), the triangulations deform infinitesimally. The departure of this chapter from previous results is that we work at any classical point and obtain results over affinoid neighborhoods, rather than just finite thickenings. Of course, we cannot prove that the triangulations vary analytically—that is patently false. Rather, our main theorem is that the *maximal non-critical parabolizations* P_x^{nc} attached to the triangulations P_x (recall §2.3.1) extend to affinoid neighborhoods of classical points on refined families of (φ, Γ)-modules.

The organization of this chapter is as follows. We first recall the recent work of Kedlaya, Pottharst and Xiao [**41**] on the Galois cohomology for (φ, Γ)-modules in families. Their main result (for us) is a finiteness result for the (φ, Γ)-cohomology over the base X. This allows us to develop a "cohomology and base change" framework.

Second, we explain the notion of a refined family of (φ, Γ)-modules and state the main theorem of this chapter. The definition is meant to capture axiomatically the *a priori* structure of Galois representations (restricted to a decomposition group at p) which appear in p-adic families of automorphic forms. We make an effort, however, to limit the axioms, forcing us to deduce certain properties which we might know *a priori* in applications. This decision has been made in order to precise the nature of variation one wants to have on hand in a p-adic family before questions about the variation of triangulations should be considered. With the definition in hand, we move towards the statement of the main theorem (Theorem 4.13).

The final section is dedicated to the proof of the theorem. It itself is broken up into three sections—each slightly more general than the previous: the non-critical case, the minimally critical case and then the general case. The non-critical case is by far the easiest to understand and the situation which we follow in spirit for the latter two proofs. We have included the middle case because there we can give a significantly less involved proof, which we hope the reader will find enlightening.

Before beginning the chapter proper, we want to especially acknowledge the debt that this work owes to the authors of [**41**]. The reader aware of their work will no doubt see their influence in the work in the below.

4.1. Galois cohomology II

Recall that in §1.2.3 we defined and studied generalized (φ, Γ)-modules Q over X. In order to focus on arithmetic aspects in the case where X is zero-dimensional, we delayed the discussion of the Galois cohomology over more general X until now.

4.1.1. Finiteness of Galois cohomology. Recall from §2.2.1 that we can associate, to a generalized (φ, Γ)-module Q, the Herr complex:

$$C^\bullet_\gamma(Q) : Q^\Delta \xrightarrow{d^1_\gamma} (Q^\Delta)^{\oplus 2} \xrightarrow{d^2_\gamma} Q^\Delta.$$

This complex depends, up to quasi-isomorphism, only on the choice of a topological generator γ of Γ/Δ where, again, Δ is the p-torsion subgroup of Γ. Just as before we define the Galois cohomology of Q as $H^i(Q) := H^i(C^\bullet_\gamma(Q))$. In order to give the important finiteness result we have to restrict Q slightly more.

Definition. *We say that a generalized (φ, Γ)-module Q is finitely presented if there exists a finite presentation*

$$M \xrightarrow{f} N \xrightarrow{g} Q \to 0$$

where M and N are (φ, Γ)-modules over X and f and g are (φ, Γ)-equivariant.

Recall that a generalized (φ, Γ)-module is a finitely presented $\mathcal{R}_X$-module with $\mathcal{R}_X$-semi-linear operations by the group Γ and a $\mathcal{R}_X$-semi-linear operator φ. Note, however, that it is not clear that a generalized (φ, Γ)-module is a finitely presented (φ, Γ)-module. In the case where $X = \mathrm{Sp}(L)$ is the affinoid spectrum of a field then it follows as in the proof Corollary 2.8 but that uses in a crucial way the structure of finitely generated modules over the Robba ring $\mathcal{R}_L$, which is not available over more general X.

Proposition 4.1. *If Q is a finitely presented generalized (φ, Γ)-module over $\mathcal{R}_X$ then $H^j(Q)$ is a finite A-module for all j.*

Proof. We thank Jay Pottharst for explaining this to us. By the main theorem of [**41**] (see the introduction of *loc. cit.*) if $Q = M$ is projective then the cohomology groups $H^\bullet(M)$ are all finite A-modules.

In general, choose a finite (φ, Γ)-equivariant presentation $M \xrightarrow{f} N \to Q \to 0$. We make use of the mapping cone of a complex [**63**, §1.5]. Since $Q \mapsto C^\bullet_\gamma(Q)$ is exact, we have a quasi-isomorphism (in the derived category of $\mathcal{R}_X$-modules)

$$C^\bullet_\gamma(Q) \overset{q}{\simeq} \mathrm{cone}\left(C^\bullet_\gamma(M) \xrightarrow{f} C^\bullet_\gamma(N)\right). \tag{4.1}$$

However, general principles say that one has a short exact sequence of complexes

$$0 \to C^\bullet_\gamma(N) \to \mathrm{cone}\left(C^\bullet_\gamma(M) \xrightarrow{f} C^\bullet_\gamma(N)\right) \to C^\bullet_\gamma(M)[-1] \to 0.$$

Thus the result follows from (4.1) and the projective case in the first paragraph. ☐

Remark. Prior to [**41**], the same finiteness result was obtained by Chenevier [**17**] under the assumption that $Q = M$ is projective and *trianguline* over X. Since our goal is to show that certain (φ, Γ)-modules are trianguline, this assumption would be too strong.

4.1.2. A homological interlude. Our next goal is to understand the Galois cohomology $H^\bullet(Q)$ in terms of the pointwise Galois cohomology groups $H^\bullet(Q_x)$. But, since we have a separate need for a base change result, we have found it most convenient to pause here and explain some formalism coming from homological algebra.

Proposition 4.2 (Künneth). *Let R be a commutative ring and $\cdots \to P^2 \to P^1 \to P^0$ be a complex of R-modules. Assume that N is an R-module such that $\mathrm{Tor}^R_j(P^p, N) = (0)$ for all $j \geq 1$ and all $p \geq 0$. Then, there is a first quadrant spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^R_p(H_q(P^\bullet), N) \Rightarrow H_{p+q}(P^\bullet \otimes_R N).$$

Proof. This is stated as [**63**, Theorem 5.6.4] under the additional hypothesis that P^p be flat over R for each $p \geq 0$. However, the proof of *loc. cit.* clearly only requires that the Tor-groups against N vanish. $\square$

We have two ways in which we will use this result.

Corollary 4.3. *Let Q be a generalized (φ, Γ)-module over X and assume that for each $x \in X$, we have $\mathrm{Tor}_j^A(Q, L(x)) = (0)$ for each $j \geq 1$. Then there is a first quadrant spectral sequence*

$$\mathrm{Tor}_p^A(H^{2-q}(Q), L(x)) \Rightarrow H^{2+p-q}(Q_x).$$

Proof. Apply Proposition 4.2 to $R = A$, the complex $P^\bullet = C_\gamma^\bullet(Q)$ and $N = L(x)$. Notice that $H_q(P^\bullet) = H^{2-q}(Q)$. $\square$

Corollary 4.4. *Let Q be a finitely presented module over $\mathcal{R}_X$. Suppose that for each $j \geq 1$ and $x \in X$ that $\mathrm{Tor}_j^A(Q, L(x)) = (0)$. Let $M \geq 1$ be an integer. Then, there is a four term exact sequence*

$$0 \to \mathrm{Tor}_2^A(Q/t^M, L(x)) \to Q[t^M] \otimes_A L(x) \to Q_x[t^M] \to \mathrm{Tor}_1^A(Q/t^M, L(x)) \to 0. \tag{4.2}$$

Proof. Let $P^\bullet$ be the two term complex $[Q \xrightarrow{t^M} Q]$. Then

$$H_q(P^\bullet) = \begin{cases} Q/t^M & \text{if } q = 0, \\ Q[t^M] & \text{if } q = 1, \text{ and} \\ 0 & \text{if } q \geq 2. \end{cases}$$

The spectral sequence from Proposition 4.2 degenerates on the E^3-page on the short exact sequence follows from the explicit terms on the E^3-page. $\square$

4.1.3. Cohomology and base change. We fix for the rest of the section a finitely presented generalized (φ, Γ)-module Q over $\mathcal{R}_X$. Suppose that $x \in X$. Notice then there are canonical maps $C_\gamma^\bullet(Q) \otimes_A L(x) \to C_\gamma^\bullet(Q_x)$ which respect the boundary maps d_γ^j. We say that $H^j(Q)$ satisfies base change if the induced map $H^j(Q) \otimes_A L(x) \to H^j(Q_x)$ is an isomorphism. Notice that since $C_\gamma^\bullet(Q)$ is concentrated in degrees $0, 1$ and 2 we have that $H^2(Q)$ always satisfies base change. In general, we can use Corollary 4.3 to understand when $H^j(Q)$ satisfies base change. For this section we denote for $0 \leq i \leq 2$ the function

$$d_i(Q, x) := \dim_{L(x)} H^i(Q_x).$$

Lemma 4.5. *Assume that for all $j \geq 1$ and $x \in X$ we have that $\mathrm{Tor}_j^A(Q, L(x)) = (0)$. If the function $d_i(Q, x)$ is locally constant with respect to x for $i \geq k$ then $H^i(Q)$ is flat on X for $i \geq k$ and satisfies base change for $i \geq k - 1$.*

Proof. Notice that since A is noetherian, reduced and each $H^i(Q)$ is finite, by Proposition 4.1, we have that $H^i(Q)$ is flat if and only $x \mapsto \dim_{L(x)} H^i(Q) \otimes_A L(x)$ is locally constant.

To check this, we apply the spectral sequence from Corollary 4.3. To help us visualize it, it is the first quadrant spectral sequence whose differentials are shown

$$
\begin{array}{llll}
0 & 0 & 0 & \cdots \\
H^0(Q) \otimes_A L(x) & \mathrm{Tor}_1^A(H^0(Q), L(x)) & \mathrm{Tor}_2^A(H^0(Q), L(x)) & \cdots \\
 & \xleftarrow{\ \partial\ } & & \\
H^1(Q) \otimes_A L(x) & \mathrm{Tor}_1^A(H^1(Q), L(x)) & \mathrm{Tor}_2^A(H^1(Q), L(x)) & \cdots \\
 & \xleftarrow{\ \partial\ } & & \\
H^2(Q) \otimes_A L(x) & \mathrm{Tor}_1^A(H^2(Q), L(x)) & \mathrm{Tor}_2^A(H^2(Q), L(x)) & \cdots
\end{array} \tag{4.3}
$$

We prove the the corollary on a case-by-case basis. Notice that if $k > 3$ then the relevant cohomologies are all zero, so the result is true. If $k = 3$ then the assumption is always true and the conclusion is that $H^2(Q)$ satisfies base change, which we remarked immediately preceeding the lemma.

Now suppose $k = 2$. Since $H^2(Q)$ satisfies base change, our assumption is that the function $x \mapsto \dim_{L(x)} H^2(Q)_x$ is locally constant. Thus $H^2(Q)$ is locally free by the first sentence of this proof. We must also show that $H^1(Q)$ satisfies base change. But, since $H^2(Q)$ is flat we know that the bottom row of (4.3) is zero except possibly in the $(0,0)$-spot. In particular, the natural map $H^1(Q) \otimes_A L(x) \to H^1(Q_x)$ is an isomorphism by Corollary 4.3. The $k = 1$ and $k = 0$ cases follow by the same arguments. $\square$

Corollary 4.6. *Suppose that for all $x \in X$, if $j \geq 1$ then $\mathrm{Tor}_j^A(Q, L(x)) = (0)$. Assume as well that the function $d_i(Q, x)$ has a local minimum at a point $x \in X$ for $i \geq k$. Then there exists an open affinoid $U \ni x$ such that $H^i(Q)$ is free on U for $i \geq k$ and $H^i(Q)$ satisfies base change on U for $i \geq k - 1$.*

Proof. By Proposition 4.1 and Nakayama's lemma, the function

$$x \mapsto \dim_{L(x)} H^2(Q) \otimes_A L(x)$$

is upper semi-continuous on X. Since $H^2(Q)$ always satisfies base change we have that $d_2(Q, x)$ is upper semi-continuous. If d_2 is minimized at x then we can find a $U_2 \ni x$ on which d_2 is constant. By Lemma 4.5, we can further assume that $H^2(Q)$ is flat and that $H^1(Q)$ satisfies base change. Using the same arguments we can further shrink X to assume that $d_1(Q, x)$ and $d_0(Q, x)$ are constant. The result then follows from Lemma 4.5 applied to $k = 0$. $\square$

Corollary 4.7. *Assume that for all $x \in X$ and $j \geq 1$ that $\mathrm{Tor}_j^A(Q, L(x)) = (0)$. Let $x \in X$ such that $H^2(Q_x) = (0)$. Moreover, assume that $Q[1/t]$ is flat over $\mathcal{R}_X[1/t]$ and $\dim_{L(x)} H^0(Q_x)$ has a local minimum at x. Then, each $H^j(Q)$ is locally free near x and satisfies base change.*

Proof. Since $H^2(Q)$ satisfies base change, Nakayama's lemma implies that we can, without loss of generality, assume that $H^2(Q) = (0)$. In that case, Proposition 2.16 implies that for all $x \in X$, that

$$\dim_{L(x)} H^1(Q_x) = \mathrm{rank}_{\mathcal{R}_{L(x)}[1/t]} Q_x[1/t] + \dim_{L(x)} H^0(Q_x).$$

The first summand on the right hand side is locally constant over X since $Q[1/t]$ is flat over $\mathcal{R}_X[1/t]$. Thus, $\dim_{L(x)} H^0(Q_x)$ is minimized if and only if $\dim_{L(x)} H^1(Q_x)$ is minimized. Our claim now follows from Corollary 4.6. $\square$

Corollary 4.8. *Suppose that $x \in X$ and if $j \geq 1$ then $\mathrm{Tor}_j^A(Q, L(x)) = (0)$. Assume that $Z \subset X$ is a Zariski dense subset and $x \in X$ such that for $0 \leq i \leq 2$,*

(a) $d_i(Q, z)$ is constant on Z, and

(b) $d_i(Q, x) = d_i(Q, z)$ for some (hence, any) points $z \in Z$.

Then, each $H^i(Q)$ is locally free near x and satisfies base change.

Proof. For $j = 2$ we have that for any $x \in X$ that

$$\dim_{L(x)} H^2(Q) \otimes_A L(x) = \dim_{L(x)} H^2(Q_x). \tag{4.4}$$

Since the left hand side is upper semi-continuous, so is the right hand. The minimum must be the constant value over Z, as Z is Zariski dense. In particular, $d_2(Q, x)$ has a local minimum at x by our assumption (b). It follows from Corollary 4.6 that $H^2(Q)$ is locally free near x and $H^1(Q)$ satisfies base change. The proof then continues by the same argument using degrees one and two in (4.4). □

4.2. Refined families

Our goal now is to state our main theorem on the variation of (φ, Γ)-modules over p-adic families. For this, we must define what we mean by a refined family of (φ, Γ)-modules. The predecessor (as far as we are concerned) to this idea is the notion of a refined family of Galois representations [**5**, Chapter 4]. For our inductive constructions it is important that we work completely in the setting of (φ, Γ)-modules which are not necessarily coming from Galois representations.

We continue to denote by X an affinoid space $\mathrm{Sp}(A)$, which we still assume is reduced. Throughout, D will denote a (φ, Γ)-module over X. We let $n := \mathrm{rank}_{\mathcal{R}_X}(D)$. If $x \in X$ then we denote by D_x the corrsponding (φ, Γ)-module over the residue field $L(x)$. It has a list of n Hodge-Tate-Sen weights $\{\kappa_1(x), \ldots, \kappa_n(x)\}$ (not in any particular order, yet). We denote by

$$X^{\mathrm{reg}} = \{x \in X : \kappa_i(x) \neq \kappa_j(x) \text{ if } i \neq j\}$$

the locus of points where D_x has regular weights. If $X_0 \subset X$ then we let $X_0^{\mathrm{reg}} := X_0 \cap X^{\mathrm{reg}}$. Finally, we recall that a subset $Z \subset X$ is said to accumulate at $x \in X$ if there exists an affinoid neighborhood basis $\{U\}_{U \in \mathcal{U}}$ of x such that $Z \cap U$ is Zariski dense in U for all $U \in \mathcal{U}$. If $Z \subset X_0 \subset X$ then we say that Z accumulates on X_0 if it accumulates at every point $x \in X_0$.

4.2.1. Statement of the main theorem.

We begin first with the definition.

Definition. *A refined family of (φ, Γ)-modules over X is*

- *A (φ, Γ)-module D over X,*
- *an ordered list of continuous characters $\delta_1, \ldots, \delta_n : \mathbf{Q}_p^\times \to A^\times$, and*
- *a Zariski dense subset $X_{\mathrm{cl}} \subset X$*

satisfying the following axioms:

(RF1) For all $x \in X$, the Hodge-Tate-Sen weights of D_x are $\{\mathrm{wt}(\delta_{1,x}), \ldots, \mathrm{wt}(\delta_{n,x})\}$. We now label the Hodge-Tate-Sen weights by $\kappa_i(x) := \mathrm{wt}(\delta_{i,x})$.

(RF2) For every $x \in X_{\mathrm{cl}}$, the characters $\delta_{i,x}$ are crystalline.

(RF3) If $x \in X_{\mathrm{cl}}$ then D_x is crystalline with Hodge-Tate weights

$$\kappa_1(x) < \cdots < \kappa_n(x),$$

(so $X_{\mathrm{cl}} \subset X^{\mathrm{reg}}$) and crystalline eigenvalues

$$\{\phi_1(x), \ldots, \phi_n(x)\} := \left\{p^{\kappa_1(x)}\delta_{1,x}(p), \ldots, p^{\kappa_n(x)}\delta_{n,x}(p)\right\}.$$

(RF4) By (RF3), every point X_{cl} has a unique triangulation P_x corresponding to the ordering $(\phi_1(x),\dots,\phi_n(x))$ of Frobenius eigenvalues. Let

$$X_{\mathrm{cl}}^{\mathrm{nc}} := \{x \in X_{\mathrm{cl}} \colon P_x \text{ is a non-critical triangulation}\}.$$

Then, if $C > 0$ the sets

$$X_{\mathrm{cl},C}^{\mathrm{nc}} := \{x \in X_{\mathrm{cl}}^{\mathrm{nc}} \colon \kappa_{i+1}(x) - \kappa_i(x) > C \text{ for } i = 1,\dots,n-1\}.$$

are Zariski dense in X and accumulate at every point in X_{cl}.

Example 4.9. The families of (φ,Γ)-modules constructed in Chapter 5 (see §5.2.1) will form refined families.

The classical example we do not cover in this work is the natural family of (φ,Γ)-modules on the Coleman-Mazur eigencurve [**20**]. In that context, the fourth axiom (RF4) follows from Coleman's control theorem [**21**].

Some remarks are in order.

- First, let us compare the definition here with the definition given in [**5**, Chapter 4]. We have put together their functions (κ_i, F_i) as well as the existence of the character $(*)$ into the characters δ_i.
- Except on the locus X_{cl}, there is no reason to see the list $\kappa_1(x),\dots,\kappa_n(x)$ as an archimedean ordering on the Hodge-Tate-Sen weights even when they are all integers and D_x is crystalline. In fact, there can be very interesting points with crystalline (φ,Γ)-modules and $\kappa_2(x) < \kappa_1(x)$. In the case of the eigencurve, these are called companion points [**7**].
- On the points $x \in X_{\mathrm{cl}}^{\mathrm{nc}}$ it follows from (RF3) and (RF2) that the parameter of P_x is $(\delta_{1,x},\dots,\delta_{n,x})$ and that $\delta_{i,x} = z^{-\kappa_i(x)}\,\mathrm{unr}(\phi_i(x))$. On the other hand, one should be careful to realize that the same is not true on all of X_{cl}. In fact, if we plug the list of eigenvalues $(\phi_1(x),\dots,\phi_n(x))$ into Proposition 2.21, it is possible that we get a parameter different than $(\delta_{1,x},\dots,\delta_{n,x})$.

Let us expand on the final remark. Suppose that $x \in X_{\mathrm{cl}}$. Then, by the regularity of the Hodge-Tate weights of D_x, and Proposition 2.21, we know that associated to the ordered list $(\phi_1(x),\dots,\phi_n(x))$ of crystalline eigenvalues there is a unique triangulation P_x of D_x. Furthermore, if we denote by $(s_1(x),\dots,s_n(x))$ the ordered list of the Hodge-Tate weights at x associated to the refinement $(\phi_1(x),\dots,\phi_n(x))$ then the dictionary of Proposition 2.21 says that the parameter of P_x is given by

$$\left(z^{\kappa_1(x)-s_1(x)}\delta_{1,x},\dots,z^{\kappa_n(x)-s_n(x)}\delta_{n,x}\right).$$

We will return to this in Proposition 4.12. Note now, however, that $s_i(x)$ not analytic in x.

In the sequel we will continue to use the notation $\phi_i(x)$ to denote the crystalline eigenvalues on X_{cl}, but be careful to realize that the function $x \mapsto \phi_i(x)$ is nothing but a mapping of sets. Indeed, we have that

$$v_p(\phi_i(x)) = \kappa_i(x) + v_p(\delta_{i,x}(p)).$$

Since $\delta_i : \mathbf{Q}_p^\times \to A^\times$ is continuous, the second summand is locally constant on X. The first summand, however, varies wildly as we just remarked, and thus $\phi_i(x)$ cannot possibly be continuous.

We now restrict our attention to a subset of the classical points.

Definition. *We say that $x \in X_{\mathrm{cl}}$ is φ-regular if*

(a) the $\phi_i(x)$ are distinct,
(b) the $\phi_i(x)$ satisfy $\phi_i(x) \neq p\phi_j(x)$ if $i < j$, and
(c) the eigenvalue $\phi_1(x)\cdots\phi_i(x)$ is multiplicity one for φ acting on $\wedge^i D_{\mathrm{cris}}(D_x)$.

Remember, this is actually the second regularity hypothesis we have put on points in X_{cl}, the first being that we have assumed that D_x has regular weight. Let us first prove a relatively easy lemma.

LEMMA 4.10. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular. Then, there exists an open affinoid $U \ni x$ such that for all $u \in U$ and all integers $r \in \mathbf{Z}$ we have*

$$H^0(\delta_{j,u}\delta_{i,u}^{-1}z^{-r}) = (0) = H^2(\delta_{j,u}\delta_{i,u}^{-1}z^{-r})$$

whenever $i \neq j$.

PROOF. Fix $\eta := \delta_j\delta_i^{-1}z^{-r}$. By Corollary 4.7 it suffices to show that $H^2(\eta_x) = (0) = H^0(\eta_x)$. However, we know that

$$\eta_x = z^{\kappa_i(x)-\kappa_j(x)-r}\,\mathrm{unr}(\phi_j(x)\phi_i(x)^{-1}).$$

The first assumption on the eigenvalues imply that $\eta_x \in \widehat{T}_g$ and thus Proposition 2.13 shows what we want. $\square$

LEMMA 4.11. *Suppose that $x \in X_{\mathrm{cl}}$ and choose a constant $C > 0$. Then, the set of points $Z := \left\{u \in X_{\mathrm{cl},C}^{\mathrm{nc}} \colon\ u \text{ is } \varphi\text{-regular}\right\}$ accumulates at x.*

PROOF. Fix constants $\nu_i = v_p(\delta_{i,x}(p))$. Since δ_i is continuous, we can find an open affinoid subset U such that $v_p(\delta_{i,u}(p)) = \nu_i$ for all $u \in U$. Notice, if $u \in X_{\mathrm{cl}} \cap U$ then

$$v_p(\phi_i(u)) = \kappa_i(u) + \nu_i. \tag{4.5}$$

Replacing C by $C = \max\{C, 1 + |\nu_i - \nu_j| : 1 \leq i,j \leq n\}$ it suffices by (RF4) to show that $X_{\mathrm{cl},C}^{\mathrm{nc}} \cap U \subset Z$. So, let us choose a point $u \in X_{\mathrm{cl},C}^{\mathrm{nc}} \cap U$ and go through the properties defining φ-regularity for u.

(a) Suppose $i \neq j$ but that $\phi_i(u) = \phi_j(u)$. Thus (4.5) implies that $\kappa_j(u) - \kappa_i(u) = \nu_i - \nu_j$. On the other hand, if $u \in X_{\mathrm{cl},C}^{\mathrm{nc}}$ then

$$C < |\kappa_j(u) - \kappa_i(u)|,$$

a contradiction.

(b) If $\phi_i(u) = p\phi_j(u)$ then we get the same contradiction.

(c) Suppose that there exists pairs of integers $i_1 < j_1, \dots, i_s < j_s$ such that $\phi_{i_1}(u)\cdots\phi_{i_s}(u) = \phi_{j_1}(u)\cdots\phi_{j_s}(u)$. By (4.5), we would get

$$sC < \sum_{k=1}^{s}(\kappa_{j_k}(u) - \kappa_{i_k}(u)) = \sum_{k=1}^{s}(\nu_{i_k} - \nu_{j_k}) \leq sC,$$

a contradiction. Now, if $\phi_1(u)\cdots\phi_i(u)$ was not a simple eigenvalue we could, after canceling, find such pairs (i_r, j_r) for some $r \leq i$.

This completes the proof. $\square$

We now fix the context in which we will work. Our next step is to construct pointwise triangulations near φ-regular classical points.

PROPOSITION 4.12. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular. Then, there exists an open affinoid $U \ni x$ such that for all $u \in U$ the (φ,Γ)-module D_u has a unique triangulation P_u with an associated parameter $(z^{m_1(u)}\delta_{1,u}, \dots, z^{m_n(u)}\delta_{n,u})$ and $m_i(u) \in \mathbf{Z}$. Further, $\sum_{j=1}^{i} m_j(u) \leq 0$ for all $1 \leq i \leq n$.*

Proof. By Lemma 4.10, we can choose U to be an affinoid open subset of X such that for all $u \in U$ and all $r \in \mathbf{Z}$ we have

$$H^0(\delta_{j,u}\delta_{i,u}^{-1}z^{-r}) = (0) \tag{4.6}$$

if $i \neq j$. Let $Z := X_{\mathrm{cl}}^{\mathrm{nc}} \cap U$. Then, the subset Z is Zariski dense in U by the axiom (RF4). Moreover, at each point $z \in Z$ we know that D_z has a (unique, by (4.6)—see the next paragraph) triangulation with ordered parameter $(\delta_{1,z}, \ldots, \delta_{n,z})$. Thus, the existence of such a triangulation is the statement of [**41**, Theorem 6.2.14]. Indeed, in their notation please take $M = D\big|_U$ and $Z = Z$. They don't explicit mention the condition on the sign of $\sum_{j=1}^i m_i(u)$ but it follows from the proof of *loc. cit.*

Suppose there exists a triangulation P'_u with parameter $(z^{m'_1(u)}\delta_{1,u}, \ldots, z^{m'_n(u)}\delta_{n,u})$ and we want to show that $m'_i(u) = m_i(u)$ for each i. For $i = 1$, we first note that (4.6) with $r = m_j(u) - m_i(u)$ says that if $j > 1$ then

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(z^{m'_1(u)}\delta_{1,u}), \mathcal{R}_L(z^{m_j(u)}\delta_{j,u})) = (0).$$

In particular, any (φ, Γ)-equivariant inclusion $\mathcal{R}_{L(u)}(z^{m'_1(u)}\delta_{1,u}) \hookrightarrow D_u$ must factor through the submodule $\mathcal{R}_{L(u)}(z^{m_1(u)}\delta_{1,u})$. Since each of these submodules are meant to be saturated inside D_u we conclude that $\mathcal{R}_{L(u)}(z^{m'_1(u)}\delta_{1,u}) \cong \mathcal{R}_{L(u)}(z^{m_1(u)}\delta_{1,u})$, i.e. $m'_1(u) = m_1(u)$. If we assume by induction that $m'_k(u) = m_k(u)$ for $k = 1, \ldots, i-1$ then we necessarily have $P_{i-1,u} = P'_{i-1,u}$. Just as before we have that if $i \neq j$ then

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(z^{m'_i(u)}\delta_{i,u}), \mathcal{R}_L(z^{m_j(u)}\delta_{j,u})) = (0).$$

So, the saturated inclusion $\mathcal{R}_{L(u)}(z^{m'_i(u)}\delta_{i,u}) \subset D_u/P_{i-1,u}$ must land inside the already saturated submodule $\mathcal{R}_{L(u)}(z^{m_i(u)}\delta_{i,u})$. Thus, $m_i(u) = m'_i(u)$. as before. □

Remark. One could also deduce the result for $x \in X_{\mathrm{cl}}^{\mathrm{nc}}$ by the proof of Theorem 4.13—investigating the proof will show that the argument would not be circular.

With the previous result in mind, we now replace X by U as in Proposition 4.12. It is clear that all the hypotheses on X still hold on U. Thus we may assume that we have associated to each point $x \in X$ a choice of triangulation which depends only on the initial data of the characters $(\delta_1, \ldots, \delta_n)$ and the set $X_{\mathrm{cl}}^{\mathrm{nc}}$.

Definition. *Let $x \in X$. Then the triangulation P_x given by Proposition 4.12 is called the canonical triangulation with respect to the parameter $(\delta_1, \ldots, \delta_n)$ and $X_{\mathrm{cl}}^{\mathrm{nc}}$. We write $(\widetilde{\delta}_{1,x}, \ldots, \widetilde{\delta}_{n,x})$ for the ordered parameter of P_x.*

Remark. We are aware that we have previously used the $\widetilde{\delta}$ notation to denote infinitesimal deformations. However, they shall never appear in this chapter and so we feel comfortable reusing the notation.

Notice that at points $x \in X_{\mathrm{cl}}$ we have an *a priori* different triangulation coming from the ordering $(\phi_1(x), \ldots, \phi_n(x))$ of the Frobenius eigenvalues. In fact, the associated parameter, by Proposition 2.21 is $(z^{-s_1(x)}\,\mathrm{unr}(\phi_1(x)), \ldots, z^{-s_n(x)}\,\mathrm{unr}(\phi_n(x)))$. However, if $x \in X_{\mathrm{cl}}$ then by (RF2) we know that $\delta_{i,x}$ is crystalline and thus of the form $z^{-\kappa_i(x)}\,\mathrm{unr}(p^{\kappa_i(x)}\delta_{i,x}(p)) = z^{-\kappa_i(x)}\,\mathrm{unr}(\phi_i(x))$. In particular, the triangulation coming from the ordering satisfies the conclusion of Proposition 4.12 and thus must itself be the canonical triangulation. In the notation above we have $m_i(x) = \kappa_i(x) - s_i(x)$. Notice, as expected, that

$$\sum_{j=1}^{i} m_j(x) = \sum_{j=1}^{i} (\kappa_j(x) - s_j(x)) \le 0.$$

Let $x \in X_{\mathrm{cl}}$ and let P_x be its canonical triangulation. We have defined in §2.3.1 what it means for a step $P_{i,x}$ to be non-critical. We now extend that to all of X.

DEFINITION. *Let $x \in X$ and let P_x be the canonical triangulation. We say that $P_{i,x}$ is non-critical if*

$$\delta_{1,x} \cdots \delta_{i,x} = \widetilde{\delta}_{1,x} \cdots \widetilde{\delta}_{i,x}.$$

We define the maximal non-critical parabolization P_x^{nc} analogous to §2.3.1: we let

$$I_x^{\mathrm{nc}} = \{i \colon P_{i,x} \text{ is non-critical}\} = \{i_1 < i_2 < \ldots < i_s\}$$

and then define P^{nc} as the parabolization

$$P_x^{\mathrm{nc}} : 0 \subsetneq P_{i_1,x} \subsetneq P_{i_2,x} \subsetneq \cdots \subsetneq P_{i_s,x} = D_x.$$

We claim that this does not overwrite the previous definition. In fact, suppose that $x \in X_{\mathrm{cl}}$. Then, since D_x has regular Hodge-Tate weights, we know that $\kappa_1(x) + \cdots + \kappa_i(x)$ is the lowest possible value for the sum of any i weights. Thus, Lemma 2.24 says that a necessary and sufficient condition for $P_{i,x}$ to be non-critical is that $s_1(x) + \cdots + s_i(x) = \kappa_1(x) + \cdots + \kappa_i(x)$. However, since $\delta_{i,x} = z^{s_i(x)-\kappa_i(x)}\widetilde{\delta}_{i,x}$, this is equivalent to the definition given above.

Further, analogous to the case of a finite $\mathbf{Q}_p$-algebra, we can define what it means to give a parabolization of a (φ, Γ)-module over X.

DEFINITION. *We say that a filtration*

$$P : 0 = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_s = D$$

is a parabolization of D if each P_i is a (φ,Γ)-module over X and such that the quotients P_i/P_{i-1} are finite projective $\mathcal{R}_X$-modules.

If P is a parabolization then since the associated gradeds are projective over X, we know that we can specialize at a point $x \in X$ and get a parabolization

$$P \otimes_A L(x) : 0 = P_0 \otimes_A L(x) \subsetneq P_1 \otimes_A L(x) \subsetneq \cdots \subsetneq P_s \otimes_A L(x) = D_x$$

over the residue field. We are now ready to state the main theorem.

THEOREM 4.13. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular. Then, there exists an open affinoid $U \ni x$ and a parabolization P^{nc} of D over U such that*

(a) $P^{\mathrm{nc}} \otimes_A L(x) = P_x^{\mathrm{nc}}$, and

(b) $P^{\mathrm{nc}} \otimes_A L(u)$ is a parabolization of D_u which is contained in P_u^{nc}.

Notice that in order to define the canonical triangulations P_u we have already "shrunk" X from the beginning of the chapter. It is entirely possible that we still must shrink X more. That is, it is not enough to just construct the pointwise triangulations.

4.2.2. Upper semi-continuity of non-critical parabolizations. Before we prove Theorem 4.13, we are going to explain a sort of upper semi-continuity of the maximal non-critical parabolizations. The result that we want to prove in this subsection is the following.

PROPOSITION 4.14. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular and suppose that $P_{i,x}$ is non-critical. Then $P_{i,u}$ is also non-critical for all u in some open affinoid neighborhood around x.*

We remark that it is a formal corollary of the theorem, but we will also use it to significantly shorten the proof of Theorem 4.13 in the minimally critical case. Further, it is psychologically important (for the author, at least) to understand this kind of basic result from the beginning. Finally, we also justify the short detour as the computations below will be used in Chapter 5.

The proof will follow from a series of lemmas. For notation, we define

$$\Delta_i := \delta_1 \cdots \delta_i.$$

This is a continuous character $\Delta_i : \mathbf{Q}_p^\times \to A^\times$. We have as well, for each $x \in X$, a continuous character

$$\widetilde{\Delta}_{i,x} := \widetilde{\delta}_{1,x} \cdots \widetilde{\delta}_{i,x} : \mathbf{Q}_p^\times \to L(x)^\times.$$

In general, we don't expect $\widetilde{\Delta}_{i,x}$ to be the evaluation of a character $\widetilde{\Delta}_i$ at x so the notation is only meant to be suggestive. But, for example we see that $P_{i,x}$ is non-critical on X if and only if $\widetilde{\Delta}_{i,x} = \Delta_{i,x}$. We first make some simple calculations at classical, non-critical points.

Lemma 4.15. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular and that $P_{i,x}$ is non-critical. Then*

$$\dim_{L(x)} H^j\left((\wedge^i D_x/t)(\Delta_{i,x}^{-1})\right) = \begin{cases} 1 & \text{if } j = 0 \text{ or } j = 1 \\ 0 & \text{if } j = 2. \end{cases}$$

and

$$\dim_{L(x)} H^j\left(\wedge^i D_x(\Delta_{i,x}^{-1})\right) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + \binom{n}{i} & \text{if } j = 1 \\ 0 & \text{if } j = 2. \end{cases}$$

and

$$\dim_{L(x)} H^j\left(t \wedge^i D_x(\Delta_{i,x}^{-1})\right) = \begin{cases} 0 & \text{if } j = 0 \\ \binom{n}{i} & \text{if } j = 1 \\ 0 & \text{if } j = 2. \end{cases}$$

Proof. Let $m := \mathrm{rank}_{\mathcal{R}_{L(x)}} \wedge^i D_x = \binom{n}{i}$. Choose any parameter $(\eta_1, \dots, \eta_m)$ of $\wedge^i D_x$ with the property that $\eta_1 = \widetilde{\delta}_{1,x} \cdots \widetilde{\delta}_{i,x} = \Delta_{i,x}$ and $\mathrm{wt}(\eta_q \Delta_{i,x}^{-1}) \geq 1$ for any $q \geq 2$. By induction and Proposition 2.14 that

$$H^0\left(\mathcal{R}_{L(x)}/t\right) \cong H^0\left((\wedge^i D_x/t)(\Delta_{i,x}^{-1})\right)$$

is one-dimensional. Together with the Euler-Poincaré-Tate characteristc formula (Proposition 2.16) we get the first computation.

We do the next two computations at the same time. We claim that that for $q \geq 2$, the characters $\eta_q \Delta_{i,x}^{-1}$ and $z\eta_q \Delta_{i,x}^{-1}$ are generic in the sense of §2.2.2. That being done, the computations follows from Lemma 2.28. We now prove the claim. Since $\mathrm{wt}(\eta_q \Delta_{i,x}^{-1}) \geq 1$ it suffices to show that neither $\eta_q \Delta_{i,x}^{-1}$ nor $z\eta_q \Delta_{i,x}^{-1}$ of the form z^{-k} for $k \geq 0$. We can actually deduce the same with any $k \in \mathbf{Z}$. So, assume that $\eta_q \Delta_{i,x}^{-1} = z^{-k}$. We see that

$$p^{\mathrm{wt}\,\eta_q} \eta_q(p) = p^{k + \mathrm{wt}\,\Delta_{i,x}} p^{-k} \Delta_{i,x}(p) = p^{\mathrm{wt}\,\Delta_{i,x}} \Delta_{i,x}(p) = \phi_1(x) \cdots \phi_i(x). \tag{4.7}$$

By the dictionary Proposition 2.21 this would imply that $\phi_1(x) \cdots \phi_i(x)$ is not a simple eigenvalue on $D_{\mathrm{cris}}(\wedge^i D_x)$, contradicting that x is φ-regular. □

Lemma 4.16. *Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular and $P_{i,x}$ is non-critical. Then, the modules $H^j(\wedge^i D/t(\Delta_i^{-1}))$, $H^j(\wedge^i D(\Delta_i^{-1}))$ and $H^j(t \wedge^i D(\Delta_i^{-1}))$ are all locally free and satisfy base change near x.*

Proof. Notice that Lemma 4.15 applies to all the φ-regular points in $X_{\mathrm{cl}}^{\mathrm{nc}}$. By Lemma 4.11, such points accumulate at x. In particular, we can apply Corollary 4.8 to deduce the result. Note that Tor-group hypothesis is valid there as $\wedge^i D$ and $\wedge^i D/t$ are each flat over X (the latter by [**41**, Corollary 2.1.7]). □

We are now in position to finish the proof of Proposition 4.14.

Proof of Proposition 4.14. Suppose that $x \in X_{\mathrm{cl}}$ is φ-regular. By Lemma 4.16 we can assume that $H^0(\wedge^i D(\Delta_i^{-1}))$ and $H^0(\wedge^i D/t(\Delta_i^{-1}))$ are each locally free of rank one and satisfy base change. Thus, there is a unique (up to $A^\times$) morphism

$$\alpha : \mathcal{R}_U(\Delta_i) \hookrightarrow \wedge^i D\big|_U \tag{4.8}$$

of (φ,Γ)-modules over U.

Let $u \in U$ and consider the specialization

$$\mathcal{R}_{L(u)}(\Delta_{i,u}) \xrightarrow{\alpha_u} \wedge^i D_u.$$

By construction we have that α_u is a non-zero element in the one-dimensional $L(u)$-vector space $\mathrm{Hom}_{(\varphi,\Gamma)}(\Delta_{i,u}, \wedge^i D_u)$. On the other hand, we also know that the submodule $\mathcal{R}_{L(u)}(\widetilde{\Delta}_{i,u}) \subset \wedge^i D_u$ is saturated. Note that by Proposition 4.12, we have $\widetilde{\Delta}_{i,u}\Delta_{i,u}^{-1} = z^{m_1(u)+\cdots+m_i(u)} \in \widehat{T}^+$. Thus, Proposition 2.13 implies that the natural inclusion

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(\Delta_{i,u}), \mathcal{R}_L(\widetilde{\Delta}_{i,u})) \hookrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_L(\Delta_{i,u}), \wedge^i D_u)$$

is an equality for dimension reasons. In particular, α_u factors through $\mathcal{R}_L(\widetilde{\Delta}_{i,u})$. In order to show $\Delta_{i,u} = \widetilde{\Delta}_{i,u}$ it now suffices to show that $\mathrm{coker}(\alpha_u)$ is free over $\mathcal{R}_{L(u)}$, i.e. $\mathrm{im}(\alpha_u)$ is saturated.

Since $H^j(t \wedge^i D(\Delta_i^{-1}))$ satisfies base change near x for each j (by Lemma 4.16 again) and vanishes for $j = 0$, we have an inclusion

$$\mathrm{Hom}_{(\varphi,\Gamma)}\left(\mathcal{R}_{L(u)}(\Delta_{i,u}), \wedge^i D_u\right) \subset \mathrm{Hom}_{(\varphi,\Gamma)}\left(\mathcal{R}_{L(u)}(\Delta_{i,u}), \wedge^i D_u/t\right).$$

Since each is one-dimensional, we have equality and we see that the composition

$$\mathcal{R}_{L(u)}(\Delta_{i,u}) \xrightarrow{\alpha_u} \wedge^i D_u \to (\wedge^i D_u)/t$$

is non-zero. Thus $\mathrm{im}(\alpha_u)$ is saturated inside $\wedge^i D_u$ by Corollary 2.8. ☐

Remark. It is important that the term $H^1(t \wedge^i D(\Delta_i^{-1}))$ satisfies base change. In fact, without that we could envision that $H^0(t \wedge^i D(\Delta_i^{-1})) = (0)$ but that $H^0(t \wedge^i D_u(\Delta_{i,u}^{-1})) \neq 0$ for some u.

4.3. Proof of the main theorem

In this section we prove Theorem 4.13, in three separate settings. We do this only as a matter of clarity, not out of logical necessity. In fact, the reader only interested in the whole proof is invited to skip ahead to §4.3.3 and read the proof there.

4.3.1. Proof of Theorem 4.13 in the non-critical case. We now prove Theorem 4.13 in the non-critical case. In the non-critical case, our proof closely follows ideas presented in [**41**, Theorem 6.2.9] but with a view towards regularity hypotheses and allowing ourselves to shrink X as necessary. This also gives a new proof of Proposition 4.12 in a neighborhood of any φ-regular non-critical point classical point.

Suppose that $x \in X_{\mathrm{cl}}^{\mathrm{nc}}$ is a φ-regular point. Let P_x be the canonical triangulation. By Proposition 4.14 we can, and do, suppose that $P_{i,u}$ is non-critical for all $u \in X$ and all $1 \leq i \leq n$.

Lemma 4.17. *There exists an open affinoid $U \subset X$ and a parabolization $P_1 \subsetneq D$ over U such that $P_1 \otimes_A L(u) \cong P_{1,u}$ for all $u \in U$.*

Proof. Since $P_{1,x}$ is non-critical, Lemma 4.16 implies that there exists a $U \ni x$ and an embedding

$$\alpha : \mathcal{R}_U(\delta_1) \hookrightarrow D\big|_U$$

such that $\operatorname{im}(\alpha_u) \not\subset tD_u$ for any $u \in U$. After possibly shrinking U further (by Lemma 4.10) we necessarily have that $\operatorname{im}(\alpha_u)$ is the saturated submodule $P_{1,u} \subset D_u$.

Consider $Q_1 := \operatorname{coker}(\alpha)$. It follows then that $Q_{1,u} = \operatorname{coker}(\alpha_u)$ is the (φ, Γ)-module $D_u/\mathcal{R}_{L(u)}(\delta_{1,u})$ over $\mathcal{R}_{L(u)}$ of rank $n-1$. By Corollary 1.4 we see that Q_1 is projective as a $\mathcal{R}_U$-module. Let $P_1 := \operatorname{im}(\alpha)$. Since Q_1 is projective, $P_1 \otimes_A L(u) \hookrightarrow D_u$ and defines a saturated submodule contained in $\operatorname{im}(\alpha_u)$. Thus, it must be equal to $\operatorname{im}(\alpha_u)$ and we have proven the statement. $\square$

Following the lemma we can now proceed to prove Theorem 4.13 by induction $\operatorname{rank}_{\mathcal{R}_X} D$. The key point here is that we have an exact sequence

$$0 \to P_1 \to D \to Q_1 \to 0$$

where Q_1 is a (φ, Γ)-module over $\mathcal{R}_X$. It is easy to see Q_1 forms a refined family over X and if x was non-critical and φ-regular with respect to D, it is still non-critical and φ-regular with respect to Q_1. Thus the induction hypothesis can be applied to Q_1 and we obtain the triangulation of D on an affinoid neighborhood of x.

4.3.2. Proof of Theorem 4.13 in the minimally critical case. With the non-critical case done, we move on to a slightly more difficult case. Note that away from the non-critical locus one sees immediately the obstruction to carrying out the proof of §4.3.1 is that one need not know that it need not be true $\operatorname{coker}(P_1 \to D)$ is projective over $\mathcal{R}_X$ (in the notation above). The next two subsections deal with how to get around that issue.

We give the proof in the special case below for two reasons. First, it gives a more concrete version of the proofs in the next section. Second, it is a uniform condition on the triangulation P_x for which the computations in §3.3.2 can be applied.

Definition. *Suppose that $x \in X_{\mathrm{cl}}$ with canonical parabolization P_x. We say that x is minimally critical if $\operatorname{rank}_{\mathcal{R}_{L(x)}} \operatorname{Gr}_j P_x^{\mathrm{nc}} \leq 2$ for each j.*

Remark. A point $x \in X_{\mathrm{cl}}$ is minimally critical if and only if $s_i(x)$ is in the set $\{\kappa_{i-1}(x), \kappa_i(x), \kappa_{i+1}(x)\}$ for each $i \geq 2$ and $s_1(x) \in \{\kappa_1(x), \kappa_2(x)\}$.

Example 4.18. Let $\operatorname{rank}_{\mathcal{R}_{L(x)}} D_x = 3$. If D_x is indecomposable, every triangulation is minimally critical (see the explanation in Proposition 3.40). On the other hand, if D_x is completely split with distinct crystalline eigenvalues then three among the six possible triangulations are minimally critical.

We now begin the proof of Theorem 4.13. We will have to redo some of the computations we had previously made but with a view towards keeping track of torsion. The following lemma will also be used in the next section (it is Lemma 4.15 with $i = 1$—the non-critical hypothesis is unnecessary in this case).

Lemma 4.19. *If $x \in X_{\mathrm{cl}}$ is φ-regular then*

$$\dim_{L(x)} H^j(D_x(\delta_{1,x}^{-1})) = \begin{cases} 1 & \text{if } j = 0, \\ n+1 & \text{if } j = 1, \\ 0 & \text{if } j = 2. \end{cases}$$

Proof. Recall that the canonical triangulation has a parameter $(\widetilde{\delta}_{1,x}, \ldots, \widetilde{\delta}_{n,x})$ such that $\widetilde{\delta}_{1,x} = z^{-m}\delta_{1,x}$ with $m \geq 0$. On the other hand, Lemma 4.10 implies that if $j > 1$ then $\widetilde{\delta}_{j,x}\delta_{1,x}^{-1}$ is generic and thus the computation follows from Lemma 2.28. $\square$

Fix now a point $x \in X_{\mathrm{cl}}$ which is φ-regular and minimally critical. Without loss of generality we can assume that at the point x we have that $P_{1,x}$ is critical but that $P_{2,x}$ is not. Indeed, if $P_{1,x}$ is non-critical then just apply Lemma 4.17 until the point where $P_{1,x}$ becomes critical. By the discussion proceeding Proposition 4.12, we have that $\widetilde{\delta}_{i,x} = z^{\kappa_i(x)-s_i(x)}\delta_{i,x}$. Thus since $P_{1,x}$ is critical but $P_{2,x}$ is not, we have that

$$\widetilde{\delta}_{1,x} = z^{\kappa_1(x)-\kappa_2(x)}\delta_{1,x}, \text{ and}$$
$$\widetilde{\delta}_{2,x} = z^{\kappa_2(x)-\kappa_1(x)}\delta_{2,x}.$$

Moreover, by the upper semi-continuity of non-critical parabolizations (Proposition 4.14), we can assume that $P_{2,u}$ is non-critical for every point $u \in X$. Thus, for every point $u \in X$ we have

$$\delta_{1,u}\delta_{2,u} = \widetilde{\delta}_{1,u}\widetilde{\delta}_{2,u}. \tag{4.9}$$

This assumption will remain in force, it will appear explicitly in the proof of Lemma 4.22.

Continuing on, Lemma 4.19 implies that the dimension of the cohomology groups $H^j(D_x(\delta_{1,x}^{-1}))$ agree with the dimension of the cohomology groups $H^j(D_u(\delta_{1,u}^{-1}))$ for all φ-regular points $u \in X_{\mathrm{cl}}^{\mathrm{nc}}$, computed by Lemma 4.15. Since such points accumulate at x, by Lemma 4.11, we deduce by Corollary 4.8 that after shrinking X we can assume that each cohomology space $H^j(D(\delta_1^{-1}))$ is free and satisfies base change. Their ranks are computed by Lemma 4.19. With this in mind, we can choose an A-basis

$$\alpha_1 : \mathcal{R}_X(\delta_1) \to D$$

for $\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_X(\delta_1), D) = H^0(D(\delta_1^{-1}))$. Denote, as before, the cokernel as Q_1. By Lemma 4.10 we may assume that $H^0(D_u(\delta_{1,u}^{-1})) \cong H^0(\widetilde{\delta}_{1,u}\delta_{1,u}^{-1})$ for all $u \in X$.

Let $u \in X$. Since $H^0(D(\delta_1^{-1}))$ satisfies base change, our previous comment implies that $\alpha_{1,u}$ is injective for all u and

$$Q_{1,u} = \mathrm{coker}\left(\mathcal{R}_{L(u)}(\delta_{1,u}) = t^{m_1(u)}\mathcal{R}_{L(u)}(\widetilde{\delta}_{1,u}) \to D_u\right).$$

Thus, Lemma 1.11 implies that we have a short exact sequence

$$0 \to \mathcal{R}_X(\delta_1) \xrightarrow{\alpha_1} D \to Q_1 \to 0. \tag{4.10}$$

Using again that $\alpha_{1,u}$ is injective for each u we deduce that

$$\mathrm{Tor}_j^X(Q_1, L(u)) = (0) \qquad (\text{if } j \geq 1). \tag{4.11}$$

Recall that we defined the integer $m_1(u)$ in Proposition 4.12. We have that $m_1(u) \leq 0$ and that $m_1(u) = 0$ if and only if $P_{1,u}$ is non-critical. In general now, for $u \in X$, we have a short exact sequence of generalized (φ,Γ)-modules over $\mathcal{R}_{L(u)}$

$$0 \to \mathcal{R}_{L(u)}(\widetilde{\delta}_{1,u})/t^{m_1(u)} \to Q_{1,u} \to D_u/P_{1,u} \to 0. \tag{4.12}$$

Consider again our fixed point $x \in X_{\mathrm{cl}}$ which is φ-regular and minimally critical. From now on, we let $M := \kappa_2(x) - \kappa_1(x) = -m_1(x)$ and consider the continuous character $z^M\delta_2 : \mathbf{Q}_p^\times \to A^\times$. Notice that

$$(z^M\delta_2)_x = z^{\kappa_2(x)-\kappa_1(x)}\delta_{2,x} = \widetilde{\delta}_{2,x}.$$

Lemma 4.20. *We have*

$$H^j(Q_{1,x}(\widetilde{\delta}_{2,x}^{-1})) = \begin{cases} 1 & \text{if } j = 0, \\ n & \text{if } j = 1, \\ 0 & \text{if } j = 2. \end{cases}$$

Proof. It suffices by the long exact sequence in cohomology attached to (4.12) at $u = x$ to show

(i) $H^0(\mathcal{R}_{L(u)}(\widetilde{\delta}_{1,x}\widetilde{\delta}_{2,x}^{-1})/t^M) = (0)$, and that

(ii) $D_u/P_{1,u}$ is almost generic in the sense of Lemma 2.28 with respect to the parmeter $(\widetilde{\delta}_{2,x}, \widetilde{\delta}_{3,x}, \dots, \widetilde{\delta}_{n,x})$.

Indeed, by (i) and local Tate duality (Proposition 2.16) We have that the canonical maps

$$H^j(Q_{1,x}(\widetilde{\delta}_{2,x}^{-1})) \to H^j(D_x/P_{1,x}(\widetilde{\delta}_{2,x}^{-1}))$$

are all isomorphisms. The dimensions then follow from (ii). To prove (i), notice that $\mathrm{wt}(\widetilde{\delta}_{1,x}\widetilde{\delta}_{2,x}^{-1}) = M$. Thus, our claim follows from Proposition 2.14. The second statement then follows because x is φ-regular (see Lemma 4.10). □

Lemma 4.21. *For each j, $H^j(Q_1(z^{-M}\delta_2^{-1}))$ is locally free near x and satisfies base change. For $j = 0$ it is locally free of rank one.*

Proof. We just computed the value of

$$u \mapsto \dim_{L(u)} H^j(Q_{1,u}(z^{-M}\delta_{2,u}^{-1})) \tag{4.13}$$

at the point $u = x$ in Lemma 4.20. Notice as well that by construction, Q_1 is a finitely presented generalized (φ, Γ)-module in the sense of Proposition 4.1. Thus, we have available the cohomology and base change formalism of §4.1.3. To finish the proof it suffices, by (4.11) and Corollary 4.8, to show that there exists a neighborhood U of x and a Zariski dense subset of points $u \in U$ for which (4.13) attains the same values as it did for x.

Consider any φ-regular point $u \in X_{\mathrm{cl}}^{\mathrm{nc}}$. The parameter of the canonical triangulation is $(\delta_{1,u}, \dots, \delta_{n,u})$. Since $P_{1,u}$ is non-critical, the short exact sequence (4.12) has no torsion submodule and $Q_{1,u} \cong D_u/P_{1,u}$. Moreover the (φ, Γ)-module $Q_{1,u}(z^{-M}\delta_{2,u}^{-1})$ is almost generic, by Lemma 4.10, with respect to the parameter $(z^{-M}, \delta_{3,u}z^{-M}\delta_{2,u}^{-1}, \dots, \delta_{n,u}z^{-M}\delta_{2,u}^{-1})$. Thus we compute its cohomology, by Lemma 2.28. Since $M \geq 0$, we get

$$\dim_{L(u)} H^0(Q_{1,u}(z^{-M}\delta_{2,u}^{-1})) = \dim_{L(u)} H^0(z^{-M}) = 1.$$

We have finished the proof. □

Following Lemma 4.21, we can shrink X and choose a basis

$$\alpha_2 : \mathcal{R}_X(z^M\delta_2) \to Q_1$$

for $\mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_X(z^M\delta_2), Q_1)$. Notice that for each $u \in X_{\mathrm{cl}}^{\mathrm{nc}}$, $\alpha_{2,u}$ is the inclusion $t^M\mathcal{R}_L(\delta_{2,u}) \hookrightarrow D_u/P_{1,u}$. Thus, by Lemma 1.11 we know that α_2 is injective. Denote the cokernel by Q_2', so that we have a short exact sequence

$$0 \to \mathcal{R}_X(z^M\delta_2) \xrightarrow{\alpha_2} Q_1 \to Q_2' \to 0. \tag{4.14}$$

What are the possible base changes of α_2? For each $u \in X$, we have a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{(\varphi,\Gamma)}\left(\mathcal{R}_{L(u)}(z^M\delta_{2,u}), \mathcal{R}_{L(u)}(\widetilde{\delta}_{1,u})/t^M\right) \to H^0(Q_{1,u}(z^{-M}\delta_{2,u}^{-1})) \to \cdots \\ \cdots \to H^0\left(D_u/P_{1,u}(z^{-M}\delta_{2,u}^{-1})\right) \to H^1\left(\mathcal{R}_{L(u)}(\widetilde{\delta}_{1,u})/t^M(z^{-M}\delta_{2,u}^{-1})\right) \to \cdots . \end{aligned}$$

attached to (4.12). Since we also know that $\dim_{L(u)} H^0(Q_{1,u}(z^{-M}\delta_{2,u}^{-1})) = 1$ and α_2 arises from base change, we get that for each u, $\alpha_{2,u}$ arises as a one of two compositions:

$$\alpha_{2,u} : \mathcal{R}_{L(u)}(z^M\delta_{2,u}) \to [Q_{1,u}]_{\mathrm{tor}} \hookrightarrow Q_{1,u}, \tag{4.15}$$

or

$$\alpha_{2,u} : \mathcal{R}_{L(u)}(z^M\delta_{2,u}) \hookrightarrow \mathcal{R}_{L(u)}(\widetilde{\delta}_{2,u}). \tag{4.16}$$

Notice that at $u = x$ we are in the second case, along with all the non-critical points (because $Q_{1,u}$ has no torsion there). In fact, we can assume that we are always in the situation (4.16). To see this, consider the sequence

$$0 \to \mathcal{R}_X(z^M\delta_2)[1/t] \xrightarrow{\alpha_2[1/t]} Q_1[1/t] \to Q_2'[1/t] \to 0$$

of $\mathcal{R}_X[1/t]$-modules. We can see explicitly that for all $u \in X$ the fiber of each term is free over $\mathcal{R}_{L(u)}[1/t]$. Thus, the function $u \mapsto \operatorname{rank}_{\mathcal{R}_{L(u)}[1/t]} Q_2'[1/t]$ is upper semi-continuous on X by Proposition 1.3 and after shrinking X we can remove the case (4.15). Actually, after knowing that the base change $\alpha_{2,u}$ is of the form (4.16) we also deduce (recall (4.11)) that

$$\operatorname{Tor}_j^X(Q_2', L(u)) = (0) \qquad (\text{if } j \geq 1). \tag{4.17}$$

We now prove the following key lemma.

Lemma 4.22. *After possibly shrinking X further, we can assume that Q_2'/t^M is free over $\mathcal{R}_X/t^M$.*

Proof. Since $H^0(Q_1(z^{-M}\delta_2^{-1}))$ is locally free of rank one and satisfies base change we see that if u is a non-critical point sufficiently close to x then there is a short exact sequence

$$0 \to \mathcal{R}_{L(u)}(\delta_{2,u})/t^M \to Q_{2,u}' \to D_u/P_{2,u} \to 0.$$

On the other hand, by construction $\alpha_{2,x}$ factors the inclusion $\mathcal{R}_{L(x)}(\widetilde{\delta}_{2,x}) \hookrightarrow D_x/P_{1,x}$:

$$\begin{array}{ccc} \mathcal{R}_{L(x)}(z^M\delta_{2,x}) & \xrightarrow{\alpha_{2,x}} & Q_{1,x} \\ \| & & \downarrow \\ \mathcal{R}_{L(x)}(\widetilde{\delta}_{2,x}) & \longrightarrow & D_x/P_{1,x} \end{array}$$

Thus, there is a short exact sequence

$$0 \to \mathcal{R}_{L(x)}(\widetilde{\delta}_{1,x})/t^M \to Q_{2,x}' \to D_x/P_{2,x} \to 0.$$

In particular, the function

$$u \mapsto \operatorname{rank}_{\mathcal{R}_{L(u)}/t} Q_{2,u}'/t \tag{4.18}$$

has a local minimum at the point $u = x$. It follows from Proposition 1.3 that after shrinking X we can assume that (4.18) is constant (and equal to $n-1$) on X. We now claim that

$$u \mapsto \operatorname{rank}_{\mathcal{R}_{L(u)}/t} Q_{2,u}'/t^M \tag{4.19}$$

is constant on X as well. Indeed, consider the base change map $\alpha_{2,u}$ again and recall that we have shown that it must be of the form (4.16). As a module over $\mathcal{R}_{L(u)}$ (but not necessarily (φ,Γ)-equivariantly) we have

$$Q_{2,u}' \cong \mathcal{R}_{L(u)}(\widetilde{\delta}_{1,u})/\mathcal{R}_{L(u)}(\delta_{1,u}) \oplus \mathcal{R}_{L(u)}(\widetilde{\delta}_{2,u})/\mathcal{R}_{L(u)}(z^M\delta_{2,u}) \oplus D_u/P_{2,u}. \tag{4.20}$$

In order for $Q_{2,u}'/t$ to be free of rank $n-1$ over $\mathcal{R}_{L(u)}/t$ we must have either $\widetilde{\delta}_{1,u} = \delta_{1,u}$ (so that the first summand disappears) or $\widetilde{\delta}_{2,u} = z^M\delta_{2,u}$ (so the second summand vanishes). Now, we finally use our assumption that every point in $u \in X$ has $P_{2,u}$ is non-critical (recall (4.9)). If $\widetilde{\delta}_{1,u} = \delta_{1,u}$ then $\widetilde{\delta}_{2,u} = \delta_{2,u}$ and we see clearly see that $Q_{2,u}'/t^M$ is free over $\mathcal{R}_{L(u)}/t^M$. On the other hand, if $\widetilde{\delta}_{2,u} = z^M \delta_{2,u}$ then $\delta_{1,u} = z^M \widetilde{\delta}_{1,u}$ and we get the same result. Now that we see that (4.19) is constant we deduce that Q_2'/t^M is (locally near x) free over $\mathcal{R}_X/t^M$ by Corollary 1.4. $\square$

We next consider the exact sequence

$$0 \to Q_2'[t^M] \to Q_2' \xrightarrow{t^M} Q_2' \to Q_2'/t^M \to 0.$$

By Lemma 4.22, the quotient is free over $\mathcal{R}_X/t^M$. In particular, since $\mathcal{R}_X/t^M$ is flat over X (by [**41**, Corollary 2.1.7]), so is Q_2'/t^M. By (4.17) and Corollary 4.4 we then deduce that the natural map $Q_2'[t^M]_u \to Q_{2,u}'[t^M]$ is an isomorphism.

Now define

$$Q_2 = \operatorname{coker}\left(Q_2'[t^M] \to Q_2'\right).$$

Note that $Q_2'[t^M]_u = \left[Q_{2,u}'\right]_{\mathrm{tor}}$ because either $\delta_{1,u} = z^M \widetilde{\delta}_{1,u}$ or $\widetilde{\delta}_{2,u} = z^M \delta_{2,u}$. Thus, we see that $Q_{2,u} \cong D_u/P_{2,u}$. In particular, $Q_{2,u}$ is finite projective over $\mathcal{R}_{L(u)}$ and of rank independent of u. We then conclude that Q_2 is finite projective over $\mathcal{R}_X$ by Corollary 1.4. Notice as well that the natural map $D \to Q_2$ is surjective as it is defined by a composition of surjections. Finally, we let P_2 denote the kernel of this natural map, i.e.

$$0 \to P_2 \to D \to Q_2 \to 0.$$

Since Q_2 is finite projective over X we deduce that

$$P_2 \otimes_X L(u) = (\ker D_u \to Q_{2,u}) = P_{2,u}.$$

Thus, again by Corollary 1.4, we deduce that P_2 is finite projective and, moreover, $P_2 \otimes_X L(u) = P_{2,u}$.

4.3.3. Proof of Theorem 4.13 at arbitrary classical points. Here we are going to give the general proof of Theorem 4.13. So, we fix a point $x \in X_{\mathrm{cl}}$ which is φ-regular. We are going to introduce (though we could've introduced it in the previous section, probably) the following notation and keep it in force throughout this section.

Definition. *Let L be a p-adic field and Q a generalized (φ, Γ)-module over $\mathcal{R}_L$. If $(Q_1, \dots, Q_s)$ is an ordered list of generalized (φ, Γ)-modules over $\mathcal{R}_L$ then we write*

$$Q \overset{\mathrm{Gr}}{\cong} \bigoplus_{i=1}^{s} Q_i$$

if the following two conditions hold:

(a) $Q \cong \bigoplus_{i=1}^s Q_i$ as an $\mathcal{R}_L$-module, and
(b) there exists a (φ, Γ)-equivariant filtration $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_s = Q$ such that

$$\mathrm{Gr}_i F_\bullet \cong Q_i,$$

for $i = 1, \dots, s$.

Example 4.23. If D_x is the (φ, Γ)-module at the point x then $D_x \overset{\mathrm{Gr}}{\cong} \bigoplus_{i=1}^s \mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x})$.

The proof of Theorem 4.13 in the most general case relies on an explicit inductive construction that will take some time to explain, but which is ultimately elementary in nature. Fix an integer r with $1 \le r \le n$ such that $P_{r,x}$ is non-critical but $P_{i,x}$ is critical for $1 \le i \le r-1$. Recall that the numbers $s_1(x), \dots, s_n(x)$ are defined as the list of weights ordered from the parameter $(\widetilde{\delta}_{1,x}, \dots, \widetilde{\delta}_{n,x})$. Out of the number r we can generate an integer $e \ge 1$ along with sequences of integers

$$1 \le j_1 < \ldots < j_{a-1} < j_a = r$$

and

$$1 = i_1 < i_2 < \cdots < i_{a-1} < i_a < r$$

by declaring (with the convention that $j_0 = 0$)

$$\begin{aligned} s_{j_{b+1}}(x) &= \min\{s_{j_b+1}(x), \dots, s_r(x)\} && (b = 0, \dots, a-1), \\ \kappa_{i_b}(x) &= s_{j_b}(x_0) && (b = 1, \dots, a). \end{aligned}$$

Notice that the final equality (resp. inequality) in the listing of the j_a (resp. i_a) follows from the choice of r. It is best to understand these sequences through an example.

Example 4.24. Let $n = 5$ and suppose $(s_i(x)) = (\kappa_5(x), \kappa_2(x), \kappa_1(x), \kappa_3(x), \kappa_4(x))$. Then, we get

$$(j_1, j_2, j_3) = (3, 4, 5) \quad \text{and} \quad (i_1, i_2, i_3) = (1, 3, 4).$$

On the other hand, if $(s_i(x)) = (\kappa_4(x), \kappa_5(x), \kappa_1(x), \kappa_3(x), \kappa_2(x))$ then

$$(j_1, j_2) = (3, 5) \quad \text{and} \quad (i_1, i_2) = (1, 2).$$

The choice of r also implies we have the following numerical relation.

Lemma 4.25. *For $b = 1, \dots, a$ we have $\kappa_{i_b}(x) \leq \kappa_{j_{b-1}}(x)$.*

Proof. Suppose that $\kappa_{i_b}(x) = s_{j_b}(x) > \kappa_{j_{b-1}}(x)$. Then, by definition of j_b we have that each of $s_{j_{b-1}+1}(x), s_{j_{b-1}+2}(x), \dots, s_r(x)$ is larger than $\kappa_{j_{b-1}}(x)$. By the choice of r, the set

$$\left\{s_{j_{b-1}+1}(x), \dots, s_r(x)\right\}$$

of $r - j_{b-1} - 1$ weights would have to come from the set

$$\left\{\kappa_{j_{b-1}+1}(x), \dots, \kappa_r(x)\right\}$$

of $r - j_b - 1$ weights. Complementary to this, the set of weights $\left\{s_1(x), \dots, s_{j_{b-1}}(x)\right\}$ is the set of lowest j_{b-1} weights $\left\{\kappa_1(x), \dots, \kappa_{j_{b-1}}(x)\right\}$. Since this implies (by Lemma 2.24) that $P_{j_{b-1},x}$ is non-critical, we contradict the definition of r. $\square$

The inductive construction we alluded to above is contained in the following proposition. After the statement, we prove the theorem and finally go back and prove the proposition. We use the convention that $i_{a+1} = r$.

Proposition 4.26. *After shrinking X there exists a sequence of quotients*

$$D \twoheadrightarrow Q_1' \twoheadrightarrow Q_1 \twoheadrightarrow \cdots \twoheadrightarrow Q_a' \twoheadrightarrow Q_a$$

of generalized (φ, Γ)-modules over X such that if $1 \leq b \leq a$ then:

(a) *For every point $u \in X$, $\mathrm{Tor}_1^X(Q_b', L(u)) = (0) = \mathrm{Tor}_1^X(Q_b, L(u))$ and*

$$Q_{b,u}' / \left[Q_{b,u}'\right]_{\mathrm{tor}} \cong Q_{b,u} / \left[Q_{b,u}\right]_{\mathrm{tor}} \cong D_u / P_{j_b,u}.$$

(b) *At the point $u = x$ we have*

$$Q_{b,x}' \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{\substack{1 \leq i \leq j_b \\ \kappa_{i_b}(x) < s_i(x)}} \mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x}) / t^{s_i(x) - \kappa_{i_b}(x)} \right) \oplus D_x / P_{j_b,x}, \text{ and}$$

$$Q_{b,x} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{\substack{1 \leq i \leq j_b \\ \kappa_{i_{b+1}}(x) < s_i(x)}} \mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x}) / t^{s_i(x) - \kappa_{i_{b+1}}(x)} \right) \oplus D_x / P_{j_b,x}.$$

(c) *Finally, for a subset of points $u \in X_{\mathrm{cl}}$ accumulating at x, we have*

$$Q'_{b,u} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{i=i_b+1}^{j_b} \mathcal{R}_{L(u)}(\delta_{i,u})/t^{\kappa_i(x)-\kappa_{i_b}(x)} \right) \oplus D_u/P_{j_b,u}, \text{ and}$$

$$Q_{b,u} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{i=i_{b+1}}^{j_b} \mathcal{R}_{L(u)}(\delta_{i,u})/t^{\kappa_i(x)-\kappa_{i_{b+1}}(x)} \right) \oplus D_u/P_{j_b,u}.$$

Moreover,

$$\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q'_{b,u}/t = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q'_{b,x}/t.$$

and

$$1 + \mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u/t} = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q_{b,x}/t.$$

Note that in (c), the xs appearing in $t^{\kappa_i(x)-\kappa_{i_b}(x)}$ are not a typo. With the proposition in mind, we give the proof of the main theorem.

Proof of Theorem 4.13. We adopt the notation of Proposition 4.26. If we take $b = a$ then $j_b = r$. Since $\kappa_{i_{a+1}} = \kappa_r$, the formulas imply that $Q_{a,u}$ is a (φ,Γ)-module of rank $n-r$ for $u = x$ and for u in a set of classical points accumulating at x. Thus, Corollary 1.4 implies that Q_a is a (φ,Γ)-module over some open affinoid subdomain $U \subset X$ containing x. If we define $P_r := \ker(D \twoheadrightarrow Q_a)$ then for each $u \in U$ we have (by (a)) a short exact sequence

$$0 \to P_r \otimes_X L(u) \to D_u \to D_u/P_{r,u} \to 0.$$

Thus $P_r \otimes_X L(u) \cong P_{r,u}$ is a (φ,Γ)-module over $\mathcal{R}_{L(u)}$ of rank r, for each $u \in U$. Again, by Corollary 1.4, we have then that P_r is a (φ,Γ)-module over U and the conditions of Theorem 4.13 are satisfied. The proof now easily follows by induction on $n = \mathrm{rank}_{\mathcal{R}_X} D$. ☐

Remark. As the reader might notice, we used all the information from the proposition at $b = a$ except the final statement about the ranks of the torsion modules. That information is only used to keep the inductive process moving along.

The rest of the section will be dedicated to giving the proof of the proposition. To simplify the exposition we are going to make a definition and then prove an easy lemma.

Definition. *Let L be a p-adic field and $\delta : \mathbf{Q}_p^\times \to L^\times$ a continuous character and $N \geq 1$ be an integer. We then define*

$$\mathrm{wt}\left(\mathcal{R}_L(\delta)/t^N\right) := \mathrm{wt}(\delta) - N.$$

If S is now a generalized torsion (φ,Γ)-module then we say that S has constant weight w if

$$S \overset{\mathrm{Gr}}{\cong} \bigoplus_{i=1}^{s} S_i$$

for torsion (φ,Γ)-modules $S_i \cong \mathcal{R}_L(\delta_i)/t^{N_i}$ of weight $w = \mathrm{wt}(S_i)$.

We should note $\mathrm{wt}\left(\mathcal{R}_L(\delta)/t^N\right)$ depends on the torsion (φ,Γ)-module $\mathcal{R}_L(\delta)/t^N$ only up to isomorphism, by Proposition 2.14. Thus the definition makes sense.

Example 4.27. If $Q'_{b,x}$ is as the statement of Proposition 4.26 then $\left[Q'_{b,x}\right]_{\mathrm{tor}}$ has constant weight $\kappa_{i_b}(x)$.

The following lemma will be used frequently in the inductive process.

Lemma 4.28. *Let L be a p-adic field and suppose that Q is a generalized (φ,Γ)-module over $\mathcal{R}_L$ whose torsion submodule Q_{tor} has constant weight w. If η is any character such that either* $\mathrm{wt}(\eta)\notin \mathbf{Z}$ *or* $\mathrm{wt}(\eta)\leq w$ *then* $H^i(Q(\eta^{-1})) = H^i(Q_{\mathrm{free}}(\eta^{-1}))$ *for all i.*

Proof. By the Euler-Poincaré-Tate characteristic formula (Proposition 2.16) for torsion modules and the definition of constant weight w, it suffices to show $H^0(S(\eta^{-1})) = (0)$ if S is a pure torsion (φ,Γ)-module of weight w. However, if $S=\mathcal{R}_L(\delta)/t^N$ with either $\mathrm{wt}(\eta)\notin\mathbf{Z}$ or $\mathrm{wt}(\eta)\leq w = \mathrm{wt}(\delta)-N$ then Proposition 2.14 verifies this. □

We now carry out the inductive construction in Proposition 4.26. The construction is by induction on a. If we take $b=0$ with $j_b=0$ then the formulas are all true with $Q_0'=Q_0=D$. Thus, that is our base case. Suppose $1\leq b\leq a$ and that we have constructed $Q_1', Q_1,\dots,Q_{b-1}',Q_{b-1}$. The following lemma gives the construction of Q_b.

Lemma 4.29. *After shrinking X, there exists generalized (φ,Γ)-modules*

$$Q_{b-1} = Q_b'^{(0)} \twoheadrightarrow Q_b'^{(1)} \twoheadrightarrow \cdots \twoheadrightarrow Q_b'^{(j_b-j_{b-1})} =: Q_b'$$

over X such that if $0\leq c\leq j_b-j_{b-1}$ then we have the following.

(a) For all $u\in X$, $\mathrm{Tor}_1^X(Q_b'^{(c)}, L(u)) = (0)$ *and*

$$Q_{b,u}'^{(c)} / \left[Q_{b,u}'^{(c)}\right]_{\mathrm{tor}} \cong D_u/P_{j_{b-1}+c,u}.$$

(b) At the point x we have

$$Q_{b,x}'^{(c)} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{\substack{i=1\\ \kappa_{i_b}(x)<s_i(x)}}^{j_{b-1}+c} \mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x})/t^{s_i(x)-\kappa_{i_b}(x)}\right) \oplus D_x/P_{j_{b-1}+c,x}.$$

(c) For a set of points $u\in X_{\mathrm{cl}}^{\mathrm{nc}}$ accumulating at x we have

$$Q_{b,u}'^{(c)} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{i=i_b+1}^{j_{b-1}+c} \mathcal{R}_{L(u)}(\delta_{i,u})/t^{\kappa_i(x)-\kappa_{i_b}(x)}\right) \oplus D_u/P_{j_{b-1}+c,u}.$$

Moreover,

$$\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}'^{(c)}/t = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q_{b,x}'^{(c)}/t - 1$$

unless $c=j_b-j_{b-1}$, in which case we have $\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}'/t = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q_{b,x}'/t$.

Proof. The proof is by induction on $c=0,1,\dots,j_b-j_{b-1}$. The case of $c=0$ is the inductive hypothesis on Q_{b-1}. So, we suppose that $c>0$ and by induction we have constructed $Q_b'^{(c-1)}$. Notice that $\left[Q_{b,x}'^{(c-1)}\right]_{\mathrm{tor}}$ is a torsion (φ,Γ)-module of constant weight $\kappa_{i_b}(x)$ in the sense above. Since $\mathrm{wt}(z^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\delta_{j_{b-1}+c,x}) = \kappa_{i_b}(x)$, Lemma 4.28 and the inductive hypothesis together imply that

$$H^i\left(Q_{b,x}'^{(c-1)}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c,x}^{-1})\right) = H^i(D_x/P_{j_{b-1}+c-1,x}(z^{\kappa_i(x)-\kappa_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c,x}^{-1})). \tag{4.21}$$

for $i=0,1,2$. Next, the (φ,Γ)-module $D_x/P_{j_b+c-1,x}$ is a trianguline (φ,Γ)-module over $\mathcal{R}_{L(x)}$ with parameter $\left(\widetilde{\delta}_{j_{b-1}+c,x},\dots,\widetilde{\delta}_{n,x}\right)$. Since x is φ-regular we see by Lemma 4.10 that the quotient

$$D_x/P_{j_{b-1}+c-1,x}(z^{\kappa_i(x)-\kappa_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c,x}^{-1})$$

is almost generic in the sense of Lemma 2.28. Thus the cohomology groups appearing in (4.21) are completely determined by

$$H^0(\widetilde{\delta}_{j_{b-1}+c,x} z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)} \delta^{-1}_{j_{b-1}+c,x}) = H^0\left(z^{\kappa_{i_b}(x)-s_{j_{b-1}+c}(x)}\right).$$

Now, recall the definition of i_b is that

$$\kappa_{i_b}(x) = \min\left\{s_{j_{b-1}}(x), \dots, s_{j_b}(x)\right\} = \min_{0\leq c\leq j_b - j_{b-1}} \left\{s_{j_{b-1}+c}(x)\right\}.$$

Thus $\kappa_{i_b}(x) - s_{j_{b-1}+c}(x) \leq 0$ and we conclude by Lemma 2.28 that

$$\dim_{L(x)} H^i\left(Q'^{(c-1)}_{b,x}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,x})\right) = \begin{cases} 1 & \text{if } i=0,\\ 1+\operatorname{rank}_{\mathcal{R}_{L(x)}} D_x/P_{j_{b-1}+c-1,x} & \text{if } i=1,\\ 0 & \text{if } i=2.\end{cases}$$

We now show that the dimensions above are the minimal possible dimensions near x by showing they agree with sufficiently general points $u \in X^{\mathrm{nc}}_{\mathrm{cl}}$ accumulating at x. Let $u \in X^{\mathrm{nc}}_{\mathrm{cl}}$ and consider the space

$$H^0\left(Q'^{(c-1)}_{b,u}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,u})\right)$$

By induction we have that

$$H^i\left(\left[Q'^{(c-1)}_{b,u}\right]_{\mathrm{free}}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,u})\right) = H^i\left(D_u/P_{j_{b-1}+c}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c-1}(x)}\delta^{-1}_{j_{b-1}+c,u})\right). \tag{4.22}$$

for $i = 0,1,2$. By Lemma 4.25 we know that $\kappa_{i_b}(x) \leq \kappa_{j_{b-1}+c}(x)$ for $c \geq 0$, with a strict inequality for $c \geq 1$. Thus, we have

$$\dim_{L(u)} H^0(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}) = 1.$$

Moreover, if $u \in X^{\mathrm{nc}}_{\mathrm{cl}}$ is φ-regular, this space determines the cohomology spaces appearing in (4.22) (just as above, via Lemma 2.28). Thus in order to show that

$$\dim_{L(x)} H^i\left(Q'^{(c-1)}_{b,x}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,x})\right) = \dim_{L(u)} H^i\left(Q'^{(c-1)}_{b,u}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,u})\right) \tag{4.23}$$

for a set of points $u \in X^{\mathrm{nc}}_{\mathrm{cl}}$ accumulating at x and $i = 0,1,2$, it suffices to show that there is some set on which we have

$$H^0\left(\left[Q'^{(c-1)}_{b,u}\right]_{\mathrm{tor}}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta^{-1}_{j_{b-1}+c,u})\right) = (0). \tag{4.24}$$

Let Z be the set of points $u \in X^{\mathrm{nc}}_{\mathrm{cl}}$ which are φ-regular and which satisfy

$$\kappa_i(u) - \kappa_{j_{b-1}+c}(u) < \kappa_{i_b}(x) - \kappa_{j_{b-1}+c}(x), \tag{4.25}$$

for $i = i_b+1, \dots, j_{b-1}+c-1$. Notice that the left hand side is negative because we take $i < j_{b-1}+c$. Thus, by Lemma 4.11 we know that Z accumulates at the point x. Next, note that if $u \in Z$ then (4.24) follows immediately from Proposition 2.14 and the inductive hypothesis

$$\left[Q'^{(c-1)}_{b,u}\right]_{\mathrm{tor}} \overset{\mathrm{Gr}}{\cong} \bigoplus_{i=i_b+1}^{j_{b-1}+c-1} \mathcal{R}_{L(u)}(\delta_{i,u}/t^{\kappa_i(x)-\kappa_{i_b}(x)}).$$

Summarizing, (4.23) holds on the set Z accumulating at x and thus the functions

$$u \mapsto \dim_{L(u)} H^i(Q_{b,u}'^{(c-1)}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c,u}^{-1})$$

have local minimums at the point $u = x$. By induction $\mathrm{Tor}_j^X(Q_b'^{(c-1)}, L(u)) = 0$ on X and thus Corollary 4.8 implies that after shrinking X (and replacing Z by its intersection, which is still accumulating at x) we can assume that

$$H^0(Q_b'^{(c-1)}(z^{\kappa_{i_b}(x)-\kappa_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c}^{-1}))$$

is free of rank one and satisfies base change. We choose a basis element

$$\alpha : t^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\mathcal{R}_X(\delta_{j_{b-1}+c}) \to Q_b'^{(c-1)}$$

for this space.

By shrinking X even further, we may also assume that the base change α_u is injective and factors the natural inclusion $\mathcal{R}_{L(u)}(\widetilde{\delta}_{j_{b-1}+c,u}) \subset D_u/P_{j_{b-1}+c-1,u}$:

$$\begin{array}{ccc} t^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\mathcal{R}_{L(u)}(\delta_{j_{b-1}+c,u}) & \xrightarrow{\alpha_u} & Q_{b,u}'^{(c-1)} \\ \downarrow & & \downarrow \\ \mathcal{R}_{L(u)}(\widetilde{\delta}_{j_{b-1}+c,u}) & \longrightarrow & D_u/P_{j_{b-1}+c-1,u} \end{array}$$

Indeed, we know by (4.21) and (4.23) that

$$\alpha_u[1/t] : \left(t^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\mathcal{R}_X(\delta_{j_{b-1}+c,u})\right)[1/t] \to Q_{b,u}'^{(c-1)}[1/t] = \left(D_u/P_{j_{b-1}+c-1,u}\right)[1/t]$$

is non-zero at $u = x$ and on the Zariski dense set Z. Moreover, $\mathrm{coker}(\alpha_u[1/t])$ is free over $\mathcal{R}_{L(u)}[1/t]$ for each u. Thus, after shrinking X again we can assume by Proposition 1.3 that this is true on all of X. The claim then follows.

With these specifications in mind we now define

$$Q_b'^{(c)} = \mathrm{coker}\left(t^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\mathcal{R}_X(\delta_{j_{b-1}+c}) \xrightarrow{\alpha} Q_b'^{(c-1)}\right).$$

Notice first that since the base change α_u is injective, it follows from induction that $\mathrm{Tor}_1^X(Q_b'^{(c)}, L(u)) = (0)$ for all u. The inductive formulas for the shape of $Q_{b,u}'^{(c)}$ are clear; for (c) please take the set Z. We just need to clarify what has happened with the ranks of $Q_{b,u}'^{(c)}/t$ for $u \in Z$ versus $u = x$. However, by construction we clearly have for any u that

$$\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}'^{(c)}/t = \mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}'^{(c-1)}/t - \varepsilon_u^{(c)}$$

where $\varepsilon_u^{(c)} \in \{0,1\}$ and $\varepsilon_u^{(c)} = 1$ if and only if $z^{\kappa_{j_{b-1}+c}(x)-\kappa_{i_b}(x)}\delta_{j_{b-1}+c,u} = \widetilde{\delta}_{j_{b-1}+c,u}$. The values of $\varepsilon_u^{(c)}$ are readily found:

(i) Suppose that $u \in Z$. Then, $\widetilde{\delta}_{j_{b-1}+c,u} = \delta_{j_{b-1}+c,u}$, since $Z \subset X_{\mathrm{cl}}^{\mathrm{nc}}$. Further, $\kappa_{i_b}(x) < \kappa_{j_{b-1}+c}(x)$ since $c > 0$ (see Lemma 4.25). In particular, $\varepsilon_u^{(c)} = 0$ for $u \in Z$ (and all c).

(ii) Now we consider $u = x$. We have $\widetilde{\delta}_{j_{b-1}+c,x} = z^{\kappa_{j_{b-1}+c}(x)-s_{j_{b-1}+c}(x)}\delta_{j_{b-1}+c,x}$. Thus,

$$\begin{aligned} \varepsilon_x^{(c)} = 1 &\iff s_{j_{b-1}+c}(x) = \kappa_{i_b}(x) \\ &\iff j_{b-1} + c = j_b \qquad \text{(by definition)} \\ &\iff c = j_b - j_{b-1}. \end{aligned}$$

From this we see that the statement about the ranks continue through the induction. ☐

We are about to complete the inductive procedure to prove Proposition 4.26, but let us just summarize where we are at. So far we have inductively constructed quotients $Q_1', Q_1, \ldots, Q_{b-1}', Q_{b-1}$ of D satisfying the hypothesis of Proposition 4.26. Further, we have explained how we can construct Q_b' out of Q_{b-1}. We now will proceed to construct Q_b out of Q_b'. The key here is going to be exploiting the structure of t-torsion in the module Q_b' or, rather, the structure of Q_b'/t over $\mathcal{R}_X/t$. At this point, we have to make a slight departure from the proof in the minimal critical case. The new step that we need is contained in the following lemma—it has nothing to do with (φ, Γ)-modules.

LEMMA 4.30. *Let Q be a finitely presented module over $\mathcal{R}_X$ and assume for each $u \in X$ and $j \geq 1$ that $\mathrm{Tor}_j^X(Q, L(u)) = (0)$. Fix a point $x \in X$ and $Z \subset X$ a subset accumulating at x. Suppose, moreover, that for all $u \in X$ we have*

$$Q_u \cong \bigoplus_{i=1}^{s} \mathcal{R}_{L(u)}/t^{m_i(u)} \oplus \mathcal{R}_{L(u)}^{\oplus r},$$

where

(a) *r and s are independent of u;*
(b) *$m_i(u) \geq 1$ is an integer for all u and $i = 1, \ldots, s$;*
(c) *$z \mapsto \min_{i=1}^{s} \{m_i(z)\}$ is constant on Z; let m be this minimum.*
(d) *$m = \min_{i=1}^{s} \{m_i(x)\}$ as well.*

Then, there exists an open affinoid $U \ni x$ such that Q/t^m is free over $\mathcal{R}_U/t^m$.

PROOF. Notice that $u \mapsto \mathrm{rank}_{\mathcal{R}_{L(u)}/t} Q_u/t$ is constant on X since s and r are independent of u. In particular, after shrinking X we can assume, by Corollary 1.4, that Q/t is free over $\mathcal{R}_X/t$. If $m = 1$ then we are done, so we should suppose that $m > 1$.

We now prove by induction on $1 < m' \leq m$ that $Q/t^{m'}$ is free over $\mathcal{R}_X/t^{m'}$. It suffices to show that, after shrinking X, $m' \leq m_i(u)$, for all i. Indeed, if that is the case then

$$Q_u/t^{m'} \cong \left(\mathcal{R}_{L(u)}/t^{m'}\right)^{\oplus s} \oplus \left(\mathcal{R}_{L(u)}/t^{m'}\right)^{\oplus r}.$$

We see then that $Q_u/t^{m'}$ is free over $\mathcal{R}_{L(u)}/t^{m'}$ for each $u \in X$ and that moreover it has constant rank. We then apply Corollary 1.4 to conclude that $Q/t^{m'}$ is free over $\mathcal{R}_X/t^{m'}$ in a neighborhood of x.

So, suppose that it is true for $m' - 1$ and we prove it for m'. Since $\mathcal{R}_X/t^{m'-1}$ is flat over X (by [**41**, Lemma 2.1.7]), we can say that the same is true for $Q/t^{m'-1}$. By hypothesis we have that $\mathrm{Tor}_j^X(Q, L(u)) = (0)$ for all $j \geq 1$ and $u \in X$. Thus, we can deduce by Corollary 4.4 that $Q_u[t^{m'-1}] \cong Q[t^{m'-1}]_u$ for all $u \in X$. Consider $Q' := \mathrm{coker}(Q[t^{m'-1}] \to Q)$. Since

$$Q[t^{m'-1}]_u = Q_u[t^{m'-1}] \cong \bigoplus_{i=1}^{s} t^{\max\{0, m_i(u) - m' + 1\}} \mathcal{R}_{L(u)}/t^{m_i(u)},$$

we see that

$$Q_u' \cong \bigoplus_{i=1}^{s} \mathcal{R}_{L(u)}/t^{\max\{0, m_i(u) - m' + 1\}} \oplus \mathcal{R}_{L(u)}^{r}, \tag{4.26}$$

for each $u \in X$. Now consider $u \in Z$ or $u = x$. By assumptions (c) and (d) we have that $m_i(u) - m' + 1 \geq m - m' + 1 \geq 1$ and thus

$$\mathrm{rank}_{\mathcal{R}_{L(u)}/t} Q_u'/t = r + s$$

for all $u \in Z$ and at $u = x$. Since Z accumulates at x, Corollary 1.4, we can shrink X so that Q'/t is free over $\mathcal{R}_X/t$. In particular then, on X we must have that $m_i(u) - m' + 1 \geq 1$ for there to be s terms in the large direct sum (4.26). That is, $m_i(u) \geq m'$ on X and we are done. ☐

With this result in hand, we now move on to finish the proof of Proposition 4.26.

LEMMA 4.31. *After shrinking X, there exists a sequence of quotients*

$$Q'_b = Q_b^{(0)} \twoheadrightarrow Q_b^{(1)} \twoheadrightarrow \cdots \twoheadrightarrow Q_b^{(i_{b+1}-i_b)} =: Q_b$$

of generalized (φ, Γ)-modules over X such that the following hold. Let $0 \leq c \leq i_{b+1} - i_b$.

(a) *For all $u \in X$, $\mathrm{Tor}_1^X(Q_b^{(c)}, L(u)) = (0)$ and $\left[Q_{b,u}^{(c)}\right]_{\mathrm{free}} = [Q'_b]_{\mathrm{free}} = D_u/P_{i_b,u}$.*

(b) *At the point x we have*

$$Q_{b,x}^{(c)} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{\substack{i=1 \\ \kappa_{i_b+c}(x) < s_i(x)}}^{j_b} \mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x})/t^{s_i(x) - \kappa_{i_b+c}(x)} \right) \oplus D_x/P_{i_b,x}.$$

(c) *On a subset of points $u \in Z \subset X_{\mathrm{cl}}^{\mathrm{nc}}$ which accumulate at x we have*

$$Q_{b,u}^{(c)} \overset{\mathrm{Gr}}{\cong} \left(\bigoplus_{i=i_b+c+1}^{j_b} \mathcal{R}_{L(u)}(\delta_{i,u})/t^{\kappa_i(x) - \kappa_{i_b+c}(x)} \right) \oplus D_u/P_{i_b,u}.$$

Moreover,

$$\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}^{(c)}/t = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q_{b,x}^{(c)}/t$$

unless $c = i_{b+1} - i_b$, in which case $\mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}^{(c)}/t - 1 = \mathrm{rank}_{\mathcal{R}_{L(x)}/t}\, Q_{b,x}^{(c)}/t$.

PROOF. We prove it by induction on $0 \leq c \leq i_{b+1} - i_b$. Notice that for $c = 0$ it is the inductive construction of $Q'_b = Q_b^{(0)}$, plus the statement about the ranks in Lemma 4.29. So, we now suppose that $0 < c \leq i_{b+1} - i_b$ and that we have constructed $Q_b^{(c-1)}$.

First, the generalized (φ, Γ)-module $Q_{b,u}^{(c-1)}/t$ is free over $\mathcal{R}_{L(u)}/t$ for any u. Further, the function

$$u \mapsto \mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}^{(c-1)}/t$$

takes the same value at $u = x$ as it does for $u \in Z$ by (c). Since Z accumulates at x we deduce that this function is minimized at $u = x$. Thus, Nakayama's lemma (Proposition 1.3 and Corollary 1.4) allows us to shrink X and assume that $Q_b^{(c-1)}/t$ is free over $\mathcal{R}_X/t$.

Second, we claim that we can as well assume that $Q_b^{(c-1)}/t^{\kappa_{i_b+c}(x) - \kappa_{i_b+c-1}(x)}$ is free over $\mathcal{R}_X/t^{\kappa_{i_b+c}(x) - \kappa_{i_b+c-1}(x)}$. To do this, we have to show that the hypotheses of Lemma 4.30 are satisfied by $Q_b^{(c-1)}$ with $m = \kappa_{i_b+c(x)} - \kappa_{i_b+c-1}(x)$. We have

(i) We have $\mathrm{Tor}_1^X(Q_b^{(c-1)}, L(u)) = 0$ for all $u \in X$ by induction.
(ii) Since the two functions $u \mapsto \mathrm{rank}_{\mathcal{R}_{L(u)}} \left[Q_{b,u}^{(c-1)}\right]_{\mathrm{free}}$ and $u \mapsto \mathrm{rank}_{\mathcal{R}_{L(u)}/t}\, Q_{b,u}^{(c-1)}/t$ are both independent of u, we have, by Corollary 2.8, a decomposition as needed in to apply the lemma. That is, (a) and (b) are true.
(iii) We just need to check the statements (c) and (d) about the minimums. At points in Z we clearly have the minimum is $\kappa_{i_b+c}(x) - \kappa_{i_b+c-1}(x)$ (remember it is the case of $c-1$ we are applying induction to). At the point $u = x$ we want to calculate

$$\min \left\{ s_i(x) - \kappa_{i_b+c-1}(x) \colon 1 \leq i \leq j_b \text{ and } \kappa_{i_b+c-1}(x) < s_i(x) \right\}.$$

But, it follows from the definitions of the i_b and j_b that this minimum is $\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)$.

Just to summarize, we have shrunk X to the point where $Q_b^{(c-1)}/t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}$ is free over $\mathcal{R}_X/t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}$. Consider the subspace $Q_b'^{(c-1)}[t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}]$. Since $Q_b^{(c-1)}$ als has no higher Tor-groups over X, Corollary 4.4 also implies that

$$Q_{b,u}^{(c-1)}[t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}] \cong Q_b^{(c-1)}[t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}]_u \tag{4.27}$$

for all $u \in X$. So, we define

$$Q_b^{(c)} := \operatorname{coker}\left(Q_b^{(c-1)}[t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}] \to Q_b^{(c-1)}\right).$$

The first thing we notice is that $\operatorname{Tor}_1^X(Q_b^{(c)}, L(u)) = (0)$ for all $u \in X$. Indeed, that follows easily from using that it is true if we replace c by $c-1$ and (4.27). We just need to check that the inductive shape of $Q_b^{(c)}$ remains.

At the points $u \in Z$ it is clear that the inductive shape remains. At the point $u = x$ we have

$$Q_{b,x}^{(c-1)}[t^{\kappa_{i_b+c}(x)-\kappa_{i_b+c-1}(x)}] \overset{\mathrm{Gr}}{\cong} \bigoplus_{\substack{i=1 \\ \kappa_{i_b+c-1}(x)<s_i(x)}}^{j_b} t^{s_i(x)-\kappa_{i_b+c}(x)}\mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x})/t^{s_i(x)-\kappa_{i_b+c-1}(x)}. \tag{4.28}$$

Suppose that $c < i_{b+1} - i_b$. By the definition of the j_b, there exists a unique $1 \le i \le j_b$ such that $s_i(x) = \kappa_{i_b+c}(x)$. Thus, the ith coordinate of (4.28) is $\mathcal{R}_{L(x)}(\widetilde{\delta}_{i,x})/t^{s_i(x)-\kappa_{i_b+c-1}(x)}$, which is the same as the ith coordinate of $Q_{b,x}^{(c-1)}$. The inductive shape of $Q_{b,x}^{(c)}$ now follows, as well as the statement about the ranks. If $c = i_{b+1} - i_b$ then the inductive shape still holds, but we realize that we didn't drop a rank and thus we get the exception in the statement of the lemma. □

CHAPTER 5

Applications to eigenvarieties

In this final chapter we explain how the previous chapters can be used to deduce new results on the local geometry of *eigenvarieties*. For the purposes of this thesis, we restrict to eigenvarieties attached to *definite unitary groups*. Our main result is that we can produce upper bounds on the Zariski tangent spaces of such families—they are evenly provably tight in significantly new cases.

Let us outline our methods. Consider an n-dimensional eigenvariety X for a definite unitary group G attached to an imaginary quadratic extension $E/\mathbf{Q}$ in which p splits (this will all be explained in §5.1). Then, at every point $x \in X$ one has associated a p-adic Galois representation $\rho_x : G_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$. For simplicity, let us assume that $\rho_x = \rho \otimes_X L(x)$ arises from a representation over the entire space X. In that case, one can consider the family of (φ,Γ)-modules $D_{\mathrm{rig}}(\rho_p)$ attached to ρ at a decomposition group above p (which is isomorphic to $G_{\mathbf{Q}_p}$). By construction, $D_{\mathrm{rig}}(\rho_p)$ will form a refined family of (φ,Γ)-modules over X. By the main result of Chapter 4, then, one knows that locally near classical points we have an analytic variation of the associated non-critical parabolizations. In particular, this is true on any infinitesimal neighborhood of a classical point.

With this in hand, we follow standard techniques which have existed since the genesis of deformation theory. The discussion above implies that formal neighborhoods of the space X naturally embed into the *paraboline deformation spaces* studied in Chapter 3 (modulo questions at a finite number of places, see §5.2.2). One can then apply Theorem 3.38 there to produce upper bounds on the size of the tangent space to X. In many cases we produce an upper bound which is also a lower bound.

The organization of the material is as follows. In the first section we explain what we mean by a eigenvariety containing an automorphic representation on a definite unitary group G. In the next section we will give a brief tour of the Galois representations attached to automorphic representations for G. The final section will contain our main results. Recall that we have chosen isomorphisms $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ and $\iota_\infty : \overline{\mathbf{Q}} \to \overline{\mathbf{C}}$ in the introduction.

5.1. Eigenvarieties for definite unitary groups

We fix $E/\mathbf{Q}$ an imaginary quadratic extension in which the prime p is split. Denote by c the non-trivial $\mathbf{Q}$-automorphism $c : E \to E$. Denote by v a fixed place above p with conjugate place, so that $p = vv^c$.

5.1.1. Definite unitary groups. Let Δ be a central simple algebra over E equipped with an involution $c : \Delta \to \Delta$ such that $c\big|_E$ is the usual complex conjugation (thus there is no confusion in notation). We also assume that Δ is unramified at the places dividing p (which is the case for all but finitely many p). We then consider the algebraic group $G_{/\mathbf{Q}}$ whose points on a $\mathbf{Q}$-algebra R are given by

$$G(R) = \left\{ X \in (\Delta \otimes_{\mathbf{Q}} R)^\times : X \cdot X^c = 1 \right\}.$$

If we write $n^2 = \dim_E \Delta$ then we call G a definite unitary group in n-variables. The real points $G(\mathbf{R})$ form a classical unitary group with some signature (r,s) and we say that G is definite if $rs = 0$. Equivalently, G is definite if and only if $G(\mathbf{R})$ is compact.

Example 5.1. The most familiar example is the case $\Delta = M_n(E)$ and c is $X \mapsto {}^tX^c$ is the usual Hermitian adjunction. The real points are the usual unitary group $\mathrm{U}(n)(\mathbf{R})$ over $\mathbf{R}$ and we denote $G = \mathrm{U}(n)$ in this case as well. It has the following local properties:

(a) For each prime ℓ which is split in E (e.g. $\ell = p$), the choice of a place $w \mid \ell$ (e.g. v) defines[1] an isomorphism $G(\mathbf{Q}_\ell) \cong_w \mathrm{GL}_n(\mathbf{Q}_\ell)$.

(b) If $n \not\equiv 2 \bmod 4$ then $G(\mathbf{Q}_\ell)$ is quasi-split for each prime ℓ.

In the case that $n \equiv 2 \bmod 4$, there is no group G satisfying the previous two properties and which is compact at infinity.

In the sequel we will be interested in the local properties of G (rather, automorphic representations for G). If ℓ is a prime we will use the notation $G_\ell := G(\mathbf{Q}_\ell)$. We denote as well the finite set of places of E at which Δ is ramified. For $G = \mathrm{U}(n)$ we have $S_\Delta = \varnothing$ and for any Δ our running assumption is that $p \notin S_\Delta$. If $\ell \notin S_\Delta$ is a prime split in E then the choice of $w \mid \ell$ determines (just as in Example 5.1) an isomorphism $G_\ell \cong_w \mathrm{GL}_n(\mathbf{Q}_\ell)$. Thus, the place w also determines subgroups

$$\begin{aligned} K_\ell &:= \mathrm{GL}_n(\mathbf{Z}_\ell), \text{ a maximal compact subgroup of } G_\ell, \\ B_\ell &:= \text{the upper triangular Borel}, \\ T_\ell &:= (\mathbf{Q}_\ell)^n = \text{ the diagonal torus, and} \\ T_{0,\ell} &:= T_\ell \cap K_\ell. \end{aligned}$$

5.1.2. Automorphic representations. For each decreasing n-tuple

$$\mathbf{k} = (k_1 \geq k_2 \geq \cdots \geq k_n)$$

of integers there is a unique irreducible complex representation of $G(\mathbf{R})$, denoted $W_{\mathbf{k}}$, of highest weight $\mathbf{k}$. If $\mathbf{k}$ is strictly decreasing then we say that $\mathbf{k}$ (or $W_{\mathbf{k}}$) is regular. Let $T \subset \mathrm{GL}_n(\mathbf{C})$ be the diagonal torus. If we fix an embedding $G(\mathbf{R}) \hookrightarrow \mathrm{GL}_n(\mathbf{C})$ (there are two, each corresponding to an embedding $E \hookrightarrow \mathbf{C}$) then the action of $G(\mathbf{R}) \cap T$ on the highest weight vector in $W_{\mathbf{k}}$ is given by the character

$$(z_1, \ldots, z_n) \mapsto \prod_{i=1}^n z_i^{k_i}.$$

Let W be any such representation. Recall that $\mathbf{A}$ (resp. $\mathbf{A}_f$) denote the ring adeles (resp. finite adeles) over $\mathbf{Q}$. We denote by $\mathcal{A}(G, W)$ the space of automorphic forms of weight W:

$$\mathcal{A}(G, W) = \left\{ G(\mathbf{A}_f) \xrightarrow{g} W^\vee \colon \begin{array}{c} f \text{ is smooth, } G(\mathbf{R})\text{-finite and} \\ f(ag) = af(g) \text{ for all } a \in G(\mathbf{Q}). \end{array} \right\}$$

The compactness of $G(\mathbf{R})$ implies that we have a decomposition

$$\mathcal{A}(G, W) = \bigoplus_{\pi_\infty = W} \pi_f^{m(\pi)},$$

where π runs over the irreducible representations of $G(\mathbf{A})$, $m(\pi)$ is a non-negative integer and $\pi = \pi_\infty \otimes \pi_f$. The representation π_f of $G(\mathbf{A}_f)$ is itself a restricted tensor product $\pi_f = \bigotimes' \pi_\ell$ over the finite primes ℓ, with each component π_ℓ being a smooth representations of $G(\mathbf{Q}_\ell)$.

Let

$$S_1 = \{\ell \colon \ell \text{ splits in } E \text{ and } \Delta \text{ is unramified at each place } w \mid \ell\}.$$

[1]The place $w \mid \ell$ fixed, we consider the embedding $j : E \to E_w \cong \mathbf{Q}_\ell$. Then we have an isomorphism $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong \mathbf{Q}_\ell^{\oplus 2}$ is given by $x \otimes a \mapsto (j(x)a, j \circ c(x)a)$. Under this identification, we see that

$$G(\mathbf{Q}_\ell) \cong \left\{(A, B) \in \mathrm{GL}_n(\mathbf{Q}_\ell)^{\oplus 2} \colon A \cdot {}^tB = 1\right\},$$

which is clearly determined by its (completely arbitrary) first coordinate A.

Fix as well a set of primes $S_0 \subset S_1$ such that S_0 has Dirichlet density one with respect to S_1. Notice then that the set $\{w\colon w \mid \ell \text{ with } \ell \in S_0\}$ has density one within the set of all places of E. Indeed, Δ ramifies at only finitely many places and the primes w of E above a split prime ℓ have density one. The set S_0 being chosen, we also introduce a compact open subgroup $K^p \subset G(\mathbf{A}_f^p)$ which we call the a tame level. Notice that at a prime $\ell \in S_0$, for each choice $w \mid \ell$ we have an isomorphism $G(\mathbf{Q}_\ell) \cong_w \mathrm{GL}_n(\mathbf{Q}_\ell)$. We assume as well that K^p is decomposed

$$K^p = K_{S_0} K^{S_0} = K_{S_0} K^{S \cup S_0} K_S \tag{5.1}$$

where

- $K_{S_0} = G(\widehat{\mathbf{Z}}_{S_0})$ is the standard maximal compact,
- S is a finite set of places (which we will assume contains p, though it does not matter here) disjoint from S_0
- if $\ell \notin S \cup S_0$ then K_ℓ is either hyperspecial compact or very special maximal compact.

If $\pi = \bigotimes' \pi_\ell$ is an automorphic representation on $G(\mathbf{A}_f)$ and π is K^p-smooth then for each $\ell \in S_0$ and each place $w \mid \ell$ we get a smooth, unramified, representation π_w of $\mathrm{GL}_n(\mathbf{Q}_\ell)$. We will briefly review what such a representation looks like in the next subsection.

5.1.3. Representations of $\mathrm{GL}_n(\mathbf{Q}_\ell)$. Throughout this section, ℓ is any prime of $\mathbf{Q}$, possibly even $\ell = p$. If $\chi : T_\ell/T_{0,\ell} \to \mathbf{C}^\times$ is a smooth character we introduce the *normalized* smooth induction

$$\mathrm{Ind}_{B_\ell}^{G_\ell} \chi = \left\{ f : G_\ell \to \mathbf{C} \colon f(bg) = \chi(b)\delta_{B_\ell}^{1/2}(b) f(g) \text{ for all } g \in G_\ell, b \in B_\ell \right\}.$$

Here, $\delta_{B_\ell} : B_\ell \to \mathbf{C}^\times$ is the modular character

$$\delta_{B_\ell}\left(\begin{pmatrix} a_1 & * & \cdots & * \\ & a_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & a_n \end{pmatrix}\right) = |a_1|_\ell^{n-1} |a_2|_\ell^{n-3} \cdots |a_n|_\ell^{1-n} = \prod_{i=1}^n |a_i|_\ell^{n-(2i-1)}.$$

It is well-known that the representation $\mathrm{Ind}_{B_\ell}^{G_\ell} \chi$ has a unique unramified subquotient $\pi(\chi)$. The assignment $\chi \mapsto \pi(\chi)$ has the properties

- $\pi(\chi) \cong \pi(\chi')$ if and only if $\chi^\sigma = \chi'$ for some permutation[2] $\sigma \in S_n$.
- $\mathrm{Ind}_{B_\ell}^{G_\ell}(\chi) = \pi(\chi)$ if and only if $\chi_i(\ell)\chi_j(\ell)^{-1} \neq \ell$ for any $i \neq j$.

Recall that the local Langlands correspondence, as resolved by Harris and Taylor for $\mathrm{GL}_n(\mathbf{Q}_\ell)$ [**35**], defines a bijection

$$\pi \overset{\mathrm{rec}}{\longleftrightarrow} (r(\pi), N(\pi))$$

between smooth irreducible representations π of G_ℓ and n-dimensional Weil-Deligne representations (r, N) of $W_{\mathbf{Q}_\ell}$. In the case that π is unramified we have

- $N(\pi) = 0$ and $r(\pi)\big|_{I_{\mathbf{Q}_\ell}} = 1$, and
- $r(\pi)(\mathrm{Frob}_\ell)$ is conjugate to $\mathrm{diag}(\chi_1(\ell), \ldots, \chi_n(\ell)) \in \mathrm{GL}_n(\mathbf{C})$.

[2]The character χ^σ is given by $(z_1, \ldots, z_n) \mapsto \chi(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$

5.1.4. Hecke algebras. The construction of eigenvarieties will depend on the choice of a Hecke algebra $\mathcal{H}$. We describe this now. The Hecke algebra is going to constructed out of local Hecke algebras at each finite prime ℓ. Here, for the first time, we begin to separate out the theory when $\ell = p$ and with $\ell \neq p$.

Let G_0 be a locally compact, totally disconnected group with left Haar measure μ. For any topological ring R we denote by

$$\mathcal{C}_c^0(G_0, R) = \{f : G_0 \to R \colon f \text{ is continuous and compactly supported}\}.$$

Suppose that $U \subset G_0$ is a compact open subgroup such that $\mu(U) = 1$ (or, rather, we renormalize $\mu(U) = 1$). Then, we define the Hecke algebra $\mathcal{H}(G_0, U) \subset C_c(G_0, \mathbf{Z})$ as

$$\mathcal{H}(G_0, U) := \left\{f \in C_c(G_0, \mathbf{Z}) \colon f(ugu') = f(g) \text{ for all } u, u' \in U\right\}.$$

This is a (not necessarily commutative) algebra under the standard convolution product. If π_0 is a smooth representation of G_0 then π_0^U is naturally a module over the Hecke algebra $\mathcal{H}(G_0, U)$. In fact, the assignment $\pi_0 \mapsto \pi_0^U$ defines an equivalence of categories between smooth representations of G_0 and $\mathcal{H}(G_0, U)$-modules.

Recall that we fixed our tame level K^p and a set of primes S_0 at the end of §5.1.2. The above discussion applies to the group $G_0 = G(\mathbf{A}_{S_0})$ with respect to its maximal compact open subgroup $U = K_{S_0} = G(\widehat{\mathbf{Z}}_{S_0})$. We define the spherical Hecke algebra

$$\mathcal{H}_{S_0}^{\mathrm{unr}} := \mathcal{H}(G(\mathbf{A}_{S_0}), G(\widehat{\mathbf{Z}}_{S_0})).$$

It is well-known that the algebra $\mathcal{H}_S^{\mathrm{unr}}$ is commutative. To recall this, denote by ϖ_{S_0} the uniformizing element of $\mathbf{A}_{S_0}$ which is the uniformizer ℓ in the ℓ-coordinate. Then, denote by x_i is the characteristic function of the double coset

$$G(\widehat{\mathbf{Z}}_{S_0}) \begin{pmatrix} I_{i\times i} & \\ & \varpi_S I_{n-i\times n-i}\end{pmatrix} G(\widehat{\mathbf{Z}}_{S_0}). \tag{5.2}$$

This is evidently an element of $\mathcal{H}_{S_0}^{\mathrm{unr}}$ and in fact there is an isomorphism

$$\mathcal{H}_{S_0}^{\mathrm{unr}} \cong \mathbf{Z}[x_0^{\pm 1}, \dots, x_{n-1}, x_n = 1].$$

The algebra $\mathcal{H}_{S_0}^{\mathrm{unr}}$ acts naturally on automorphic representations π of tame level K^p. If π is a representation with tame level K^p then $\pi^{G(\widehat{\mathbf{Z}}_{S_0})}$ is necessarily one-dimensional and thus π defines a character $\mathcal{H}_{S_0}^{\mathrm{unr}} \to \mathbf{C}$. We can post-compose with with $\iota_p\iota_\infty^{-1}$ to get a character

$$\psi_{\pi,\mathrm{unr}} : \mathcal{H}_{S_0}^{\mathrm{unr}} \to \overline{\mathbf{Q}}_p.$$

We also must specify a local Hecke algebra at the prime p. Recall that we have assumed that p is split in E and Δ is unramified at p. We have also fixed our choice of $v \mid p$. Thus we have an isomorphism $G(\mathbf{Q}_p) \cong_v \mathrm{GL}_n(\mathbf{Q}_p)$. The Iwahori subgroup $I \subset G(\mathbf{Q}_p)$ is defined to be the subgroup

$$I = \left\{ g \in \mathrm{GL}_n(\mathbf{Z}_p) \colon g \equiv \begin{pmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \vdots \\ & & & * \end{pmatrix} \bmod p \right\}.$$

This is the higher-dimensional analog of the classical congruence subgroup $\Gamma_0(p) \subset \mathrm{SL}_2(\mathbf{Z})$. It is a compact open subgroup and we can consider the Hecke algebra $\mathcal{H}(G(\mathbf{Q}_p), I)$. Unfortunately, it is not necessarily commutative. To that end, consider the characteristic functions $[Iu^-I]$ of double cosets Iu^-I where u^- lies in the submonoid $U^- \subset T_p$ given by

$$U^- = \left\{ \begin{pmatrix} p^{m_1} & & & \\ & p^{m_2} & & \\ & & \ddots & \\ & & & p^{m_n} \end{pmatrix} : m_1 \geq m_2 \geq \cdots \geq m_n \right\}.$$

These elements of $\mathcal{H}(G(\mathbf{Q}_p), I)$ commute with each other and are invertible inside $\mathcal{H}(G(\mathbf{Q}_p), I)$ (though not within U^-). We define the Atkin-Lehner algebra $\mathcal{A}_p$ as the subalgebra generated by $\{[Iu^-I]\}_{u^- \in U^-}$, together with their inverses. In fact, if U denotes the group generated by U^- then $\mathcal{A}_p \cong \mathbf{Z}[U]$.

The algebra $\mathcal{A}_p$ again arises naturally as the domain for certain characters attached to automorphic representations unramified at p. Indeed, suppose that π_p is an unramified representation of G_p and fix a smooth character $\chi : T_p/T_{0,p} \to \mathbf{C}^\times$ such that $\pi_p \cong \pi(\chi)$. We have a name for such characters.

Definition. *A refinement of π_p is the choice of a character such that $\pi_p \cong \pi(\chi)$.*

Let us return for a moment to the normalized induction $\mathrm{Ind}_{B_p}^{G_p} \chi$ and consider the $\mathcal{H}(G_p, I)$-module of Iwahori-fixed vectors $\mathrm{Ind}_{B_p}^{G_p}(\chi)^I$. It is well-known (see [**5**, §6.4.4], for example) that we have a decomposition

$$\mathrm{Ind}_{B_p}^{G_p}(\chi)^{I,\mathcal{A}_p-\mathrm{ss}} \cong \bigoplus_{\sigma \in S_n} \chi^\sigma \delta_{B_p}^{-1/2}. \tag{5.3}$$

The action of $\mathcal{A}_p$ on the right hand side is through the natural action of $U \cong T_p/T_{0,p}$.

Now assume that χ is a refinement for π_p. We can consider the Iwahori fixed vectors $\pi(\chi)^I$. Its $\mathcal{A}_p$-semi-simplificiation embeds into the left hand side of (5.3).

Definition. *An accessible refinement is a character χ_0 such that $\chi_0\delta_B^{-1/2}$ appears in the induced decomposition $\pi(\chi)^{I,\mathrm{ss}}$.*

We have some short remarks. First, any such character is of the form χ^σ for $\sigma \in S_n$. Thus, an accessible refinement χ^σ is actually a refinement in the sense that $\pi_p \cong \pi(\chi) \cong \pi(\chi^\sigma)$. Second, since π_p is smooth, the space π_p^I is finite-dimensional and thus any character $\chi^\sigma\delta_B^{-1/2}$ which appears in the decomposition of $\pi_p^{I,\mathrm{ss}}$ is also a subrepresentation of π_p^I. The third thing is that, as we have mentioned, unless $\chi_i(p)\chi_j(p)^{-1} = p$ for some $i \neq j$ we have that $\mathrm{Ind}_B^G(\chi)$ is itself unramified. Thus, except in a special case, every refinement χ^σ defines an accessible refinement.

As $U \cong T_p/T_{0,p}$, a refinement χ determines a character $\mathcal{A}_p \cong \mathbf{Z}[U] \xrightarrow{\chi} \mathbf{C}$. Again, by post-composing with $\iota_p\iota_\infty^{-1}$ we see it has a character $\mathcal{A}_p \to \overline{\mathbf{Q}}_p$. Note that the choice of χ matters, i.e. replacing χ by χ^σ will change the character. We will come back to this point shortly (and renormalize the character itself, hence not naming it yet).

5.1.5. Galois representations. We describe here the Galois representations attached to automorphic forms π on G. Choose an automorphic representation π for G and assume that π_∞ has regular weight, π_p is unramified and π^p has level K^p. Recall again our decomposition (5.1) and specifically the finite set of primes S. If it offers no confusion, we denote by S as well the set of primes of E above a prime in S, together with the two places dividing p.

We offer the following as both a proposition and also an assumption. After making the statement we will go to some length to outline the known cases as can be best deduced from the literature. If

$\rho : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is a representation then we let $\rho^\perp$ be the representation $g \mapsto \rho^\vee(cgc)$, where $\rho^\vee$ is the usual linear dual.

Proposition/Assumption 5.2. *There exists a unique, continuous, semi-simple representation*

$$\rho_\pi : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

satisfying the following three properties.

(a) *ρ_π is conjugate self-dual: $\rho_\pi(n-1) \cong \rho_\pi^\perp$.*

(b) *Assume that $\ell \in S_0$. Choose a place $w \mid \ell$. If π is unramified at ℓ then ρ_π is unramified at w and $\mathrm{WD}(\rho_\pi|_{E_w})^{F-\mathrm{ss}} = \iota_p\iota_\infty^{-1}\operatorname{rec}(\pi_w\,|\det|^{\frac{1-n}{2}})$.*

(c) *The representation ρ_π is crystalline at v. The Hodge-Tate weights are given by*

$$-k_1 < -k_2 + 1 < \cdots < -k_n + n - 1,$$

and $\mathrm{WD}(\rho_\pi|_{E_v})^{F-\mathrm{ss}} = \iota_p\iota_\infty^{-1}\operatorname{rec}(\pi_v\,|\det|^{\frac{1-n}{2}})$.

Before remarking on how to deduce (that which can be deduced) the result from the literature, let us make some undeniable claims. First, the uniqueness of ρ_π follows from the continuity, semi-simplicity and property (b). Indeed, the primes of E above S_0 have Dirichlet density one and so this claim follows from Cebotarev. Second, (a) actually follows from (b) by considering the two places w and w^c dividing a prime $\ell \in S_0$ and basic properties of the local Langlands correspondence. Finally notice as well that (a) implies

$$\rho_\pi|_{G_{E_{v^c}}} \cong \rho_\pi^\vee(1-n)|_{G_{E_v}}.$$

Thus it follows from the third condition that ρ_π is crystalline at v^c as well. The reader is welcome to work out its Hodge-Tate weights and the characteristic polynomial of its crystalline Frobenius.

Let us now remark on the proof. There are two separate issues: Langlands base change between G and $\mathrm{GL}_{n/E}$ and the construction of Galois representations for automorphic representations over GL_n. Let us focus on the second point first.

Denote by Π an automorphic representation of $\mathrm{GL}_{n/E}$. We assume that Π is regular, cuspidal, algebraic and *essentially self-dual*: $\Pi^c \cong \Pi^\vee$. The Galois representation corresponding to Π was then constructed by Shin [**60**, Theorem 1.2] under the extra hypothesis that Π be Shin-regular (which means regular if n is odd and something slightly stronger if n is even). For Shin-regular Π, the local-global compatibility at primes away from p and the crystalline property at the primes dividing p were proven in *loc. cit.* At the primes dividing p, the local-global compatibility was proven by Barnet-Lamb, Gee, Geraghty and Taylor [**1**]. The constructions of the Galois representations for non-Shin-regular Π is obtained via an eigenvariety argument[3] due to Chenevier and Harris [**18**]. A weak form of local-global compatibility away from p and, again, the crystalline property at the primes dividing p is also proved in *loc. cit.* The strong form of local-global compatibility away from p in these cases was proven by Caraiani [**13**, Theorem 1.1]. The full local-global compatibility at places dividing p in the non-Shin-regular case is also due to Caraiani [**14**], though for the the crystalline case we restrict ourselves to on the group G it would be enough to use [**1**, Theorem A].

The question remains as to when one can use the previous paragraph to construction the representations given in Proposition 5.2. This is a question of functoriality in the Langlands program—specifically, base change. The situation here is slightly more dire than the above. In fact, Proposition 5.2 can, at the moment, only be deduced in the following two known situations:

- The explicit work of Rogawski [**54**] settles the cases of $n \leq 3$.
- If $G(E_w)$ is a central division algebra at a finite place w then he follows from work of Harris and Labesse [**34**]. Note that this condition rules out the groups $\mathrm{U}(n)$.

[3]Using the same eigenvarieties we are attempting to explain!

There is more to say, however. Had we worked with a general CM extension E/F and G a definite unitary group over F attached to this extension then all instances of base change are known *in the case that* $F \neq \mathbf{Q}$ by [**44**]. One should note that the footnote on the first page of *loc. cit.* suggests that the case of $F = \mathbf{Q}$ will soon be treated as well. In summary, we know Proposition 5.2 for sure in the two cases bulleted above, but the general case is probably within reach of current technology.

Let us finally point out that Proposition 5.2 makes no claim about the irreducibility of ρ_π. This has nothing to do with base change and everything to do with questions over $\mathrm{GL}_{n/E}$. In general, one expects that the representations attached to cuspidal Π will be irreducible, though this seems to still be unknown. For a result in this direction, however, we note that a recent preprint of Patrikis and Taylor announces a result [**52**, Theorem D] that if we fix π with cuspidal base change Π then there are infinitely p for which ρ_π is irreducible.

5.1.6. The relationship between automorphic and Galois data. In order to bring together the p-adic interpolation of automorphic representations with the p-adic interpolation of Galois representations, we would do well to remind ourselves what the local-global compatibility statements (b) and (c) of Proposition 5.2 mean. Specifically, we want to relate ρ_π to the characters of the Hecke algebra described in §5.1.4. Let π be as above and fix a prime ℓ (including the possibility that $\ell = p$ for the moment) so that π_ℓ is unramified and ℓ is split in E. Choose a place $w \mid \ell$ (taking $w = v$ if $\ell = p$).

If $\ell \neq p$ then the local-global compatibility statement Proposition 5.2(b) is equivalent to $\rho_\pi(\mathrm{Frob}_w)$ and $\iota_p^{-1}\iota_\infty r(\pi_w\,|\det|^{\frac{1-n}{2}})(\mathrm{Frob}_w)$ having the same characteristic polynomials. In the case that $\ell = p$ Proposition 5.2(c) is equivalent to the operator $\iota_p^{-1}\iota_\infty r(\pi_v\,|\det|^{\frac{1-n}{2}})(\mathrm{Frob}_v)$ having the same characteristic polynomial as the crystalline Frobenius φ acting on $D_{\mathrm{cris}}(\rho_\pi|_{E_v})$. We now compute the automorphic side of these relations.

We first concentrate on $\ell \neq p$. Choose a smooth unramified character $\chi : T_\ell \to \mathbf{C}^\times$ such that $\pi_w \cong \pi(\chi)$. We consider the character χ as a product of n characters $\chi = \prod_{i=1}^n \chi_i$ with $\chi_i : \mathbf{Q}_\ell^\times \to \mathbf{C}^\times$, trivial on $\mathbf{Z}_\ell^\times$. Then, as remarked earlier we have

$$r(\pi_w)(\mathrm{Frob}_\ell) \sim \mathrm{diag}\left(\chi_1(\ell), \dots, \chi_n(\ell)\right).$$

Thus, we see that

$$\begin{aligned} r(\pi_w\,|\det|^{\frac{1-n}{2}})(\mathrm{Frob}_\ell) &= \left(r(\pi_w) \otimes |{-}|^{\frac{1-n}{2}} \circ \mathrm{Art}_\ell\right)(\mathrm{Frob}_\ell) \\ &\sim \mathrm{diag}\left(\ell^{\frac{n-1}{2}}\chi_1(\ell), \dots, \ell^{\frac{n-1}{2}}\chi_n(\ell)\right). \end{aligned} \tag{5.4}$$

On the other hand, as explained in §5.1.4, the representation π_w defines a character

$$\mathcal{H}(G(\mathbf{Q}_\ell), G(\mathbf{Z}_\ell)) \cong \mathbf{Z}[x_0^{\pm 1}, \dots, x_{n-1}] \overset{\psi_{\pi_w,\mathrm{unr}}}{\longrightarrow} \overline{\mathbf{Q}}_p. \tag{5.5}$$

The space $\pi_w^{G(\mathbf{Z}_\ell)}$ is one-dimensional and one can explicitly compute the action of x_i on a basis element using the double coset (5.2). What one discovers[4] is that

$$\sum_{i=0}^{n}(-1)^{n-i}\psi_{\pi_w,\mathrm{unr}}(x_i)X^i = \prod_{i=1}^{n}(X - \ell^{\frac{n-1}{2}}\chi_i(\ell)). \tag{5.6}$$

Combining (5.4), (5.6) and Proposition 5.2(b), we see that the character $\psi_{\pi_w,\mathrm{unr}}$ encodes the same information as the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_w)$.

[4] Let us just give an example, with $n = 3$ and $i = 2$. Then, we have a decomposition

$$\mathrm{GL}_3(\mathbf{Z}_\ell)\begin{pmatrix}1&&\\&1&\\&&\ell\end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_\ell) = \begin{pmatrix}1&&\\&1&\\&&\ell\end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_\ell) \sqcup \coprod_{b=0}^{\ell-1}\begin{pmatrix}1&&\\&\ell&b\\&&1\end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_\ell) \sqcup \coprod_{b,c=0}^{\ell-1}\begin{pmatrix}\ell&b&c\\&1&\\&&1\end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_\ell)$$

Now suppose that $\ell = p$. The computation (5.4) of the Weil-Deligne representation is still fine, but we will push further in our interpretation on the Hecke side. Recall that the Atkin-Lehner algebra $\mathcal{A}_p$ is the subalgebra of $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p), I)$ generated by the group U. Let χ be an accessible refinement π_v. Recall that in our discussion at the end of §5.1.4 we explained how the character χ determines a character $\chi : \mathcal{A}_p \to \overline{\mathbf{Q}}_p$, because $U \cong T_p/T_{0,p}$. We now normalize this character by the following formula

$$\psi_{\pi_v,p,\chi} := \chi\delta_B^{-1/2}\delta_{\mathbf{k}}, \tag{5.7}$$

where $\delta_{\mathbf{k}} : T_p \to \overline{\mathbf{Q}}_p^\times$ is given by

$$\delta_{\mathbf{k}}\left(\mathrm{diag}(z_1, \ldots, z_n)\right) := z_1^{k_1}\cdots z_n^{k_n}. \tag{5.8}$$

We introduce elements

$$u_i := [Ix_{i-1}I][Ix_iI]^{-1} \in \mathcal{A}_p. \tag{5.9}$$

for $i = 1, \ldots, n$. Passing from the left hand side to the right hand side of (5.3) we see that u_i acts on $\mathrm{Ind}_B^G(\chi)^{I,\mathrm{ss}}$ through the value of the matrix

$$u_i = \begin{pmatrix} I_{i-1\times i-1} & & \\ & p & \\ & & I_{n-i}\times I_{n-i}\end{pmatrix} \in T \subset \mathcal{A}_p^\times.$$

on the various characters $\chi^\sigma\delta_B^{-1/2}$.

We can then easily compute the value of $\psi_{\pi_w,p,\chi}$ on the elements u_i:

$$\begin{aligned}\psi_{\pi_v,p,\chi}(u_i) &= \chi_i(p)\,|p|^{-\left(\frac{n+1-2i}{2}\right)}\,p^{k_i}\\ &= \chi_i(p)p^{\frac{n-1}{2}+k_i-i+1}.\end{aligned} \tag{5.10}$$

Comparing this with (5.4) and Proposition 5.2(c), we see that the action of the crystalline Frobenius φ on $D_{\mathrm{cris}}(\rho_\pi|_{G_{E_v}})$ has characteristic polynomial

$$\det\left(X - \varphi|_{D_{\mathrm{cris}}(\rho_\pi|_{G_{E_v}})}\right) = \prod_{i=1}^n\left(X - p^{\frac{n-1}{2}}\chi_i(p)\right) = \prod_{i=1}^n\left(X - p^{-k_i+i-1}\psi_{\pi_v,p,\chi}(u_i)\right).$$

In particular, the crystalline eigenvalues for ρ_π are the numbers $p^{\kappa_{i,\pi}}\psi_{\pi,p,\chi}(u_i)$ where $\kappa_{i,\pi}$ is the ith Hodge-Tate weight of ρ_π (when ordered in increasing order). Moreover, the choice of the accessible refinement χ determines an ordering

$$\left(p^{-k_1}\psi_{\pi,p,\chi}(u_1), p^{-k_2+1}\psi_{\pi,p,\chi}(u_2), \ldots, p^{-k_n+n-1}\psi_{\pi,p,\chi}(u_n)\right) \tag{5.11}$$

Thus if f is a basis of $\pi_w^{G(\mathbf{Z}_\ell)} \cong \mathrm{Ind}_B^G(\chi)^{\mathrm{GL}_3(\mathbf{Z}_\ell)}$ then it is determined by $f(1)$. Recall that $\delta_B = \mathrm{diag}(|-|, 1, |-|^{-1})$ in this case. Then

$$\begin{aligned}T_1f(1) &= f\left(\begin{pmatrix}1 & & \\ & 1 & \\ & & \ell\end{pmatrix}\right) + \sum_{b=0}^{\ell-1} f\left(\begin{pmatrix}1 & & \\ & \ell & b\\ & & 1\end{pmatrix}\right) + \sum_{b,c=0}^{\ell-1} f\left(\begin{pmatrix}\ell & b & c\\ & 1 & \\ & & 1\end{pmatrix}\right)\\ &= \left(\chi_3(\ell)\,|\ell|^{-1} + \sum_{b=0}^{\ell-1}\chi_2(\ell)\,|\ell|^0 + \sum_{b,c=0}^{\ell-1}\chi_1(\ell)\,|\ell|^{-1}\right)f(1)\\ &= \ell(\chi_1(\ell) + \chi_2(\ell) + \chi_3(\ell))f(1).\end{aligned}$$

of the crystalline eigenvalues. We end with a short remark on the regularity of the eigenvalues. Recall that if $\pi_v \cong \pi(\chi)$ with $\chi_i(p)\chi_j(p)^{-1} \neq p$ then every refinement is accessible and $\pi_v \cong \mathrm{Ind}_B^G(\chi)$. In particular, the previous discussions says that every refinement is accessible if and only if $\phi \neq p\phi'$ for any choice of crystalline eigenvalues ϕ and ϕ'.

5.1.7. Eigenvarieties. Our final preparation is to describe what we mean by an eigenvariety and give its main properties. The statement of our theorems on eigenvarieties will wait until §5.3.

Throughout we will denote by $\mathcal{W}$ the rigid analytic weight space

$$\mathcal{W} := \mathrm{Hom}_{\mathrm{cont}}\left((\mathbf{Z}_p^\times)^n, \mathbf{G}_m^{\mathrm{rig}}\right).$$

The $\mathbf{C}_p$-points are a disjoint union of polydiscs. It contains a collection of points $\mathbf{Z}^n$ given by

$$(k_1, \dots, k_n) \leftrightarrow \Big((z_1, \dots, z_n) \mapsto z^{k_1} \cdots z^{k_n}\Big)$$

In particular, if $\mathbf{k}$ is a highest weight for a $G(\mathbf{R})$-representation then sequence $\mathbf{k} \in \mathbf{Z}^n$ corresponds to the point $\delta_{\mathbf{k}} \in \mathcal{W}(\mathbf{Q}_p)$ we defined in (5.8). We will make use of a standard notation

$$\mathbf{Z}^{n,-} = \{\mathbf{k} \in \mathbf{Z}^n \colon k_1 > k_2 > \cdots > k_n\}.$$

The set $\mathbf{Z}^{n,-}$ is Zariski dense in $\mathcal{W}$. Moreover, each subset of the form

$$\mathbf{Z}_c^{n,-} = \left\{(k_1, \dots, k_n) \in \mathbf{Z}^{n,-} \colon k_i - k_{i+1} > c \text{ for } i = 1, \dots, n-1 \text{ and } k_n > c\right\}$$

is Zariski dense and accumulates at every point in $\mathbf{Z}^n$.

Consider automorphic representations π for G with regular weight and such that π_p is unramified. We recall that we have defined our set S_0 and the unramified Hecke algebra $\mathcal{H}_{S_0}^{\mathrm{unr}}$. Assume that π is unramified on S_0. In that case we have a character $\psi_{\pi,\mathrm{unr}} : \mathcal{H}_{S_0}^{\mathrm{unr}} \to \overline{\mathbf{Q}}_p$ by (5.5). Consider as well the choice of an accessible refinement χ for π_v. Recall as well that χ defines a character $\psi_{\pi,p,\chi}$ by the formula (5.7). Thus, if we define $\mathcal{H} = \mathcal{A}_p \otimes \mathcal{H}_{S_0}^{\mathrm{unr}}$ then we get a character

$$\psi_{\pi,\chi} : \mathcal{H} \to \overline{\mathbf{Q}}_p$$

given by $\psi_{\pi,\chi} = \psi_{\pi,p,\chi} \otimes \psi_{\pi,\mathrm{unr}}$. Define now a set

$$\mathcal{Z}_{\mathrm{cl}} := \{(\psi_{\pi,\chi}, \mathbf{k}_\pi)\}_{(\pi,\chi)} \subset \mathrm{Hom}(\mathcal{H}, \overline{\mathbf{Q}}_p) \times \mathbf{Z}^{n,-},$$

where (π, χ) runs through the possible choices above.

Remark. We want to make a quick remark on the choice of the Hecke algebra $\mathcal{H}$. In particular, we keep reminding the reader that the primes above S_0 have density one among all the primes of E but seen as a set of primes of $\mathbf{Q}$ we get at most half the primes. However, if we fix π as above and assume that it has tame level K^p decomposed as in (5.1) then we could as well consider any commutative algebra $\mathcal{H}^{S_0 \cup \{p\}} \subset \mathcal{H}(G(\mathbf{A}^{S_0}), K^{S_0})$ and study characters $\mathcal{H} \otimes \mathcal{H}^{S_0} \to \overline{\mathbf{Q}}_p$.

However, for arithmetic information our specifications are sufficient. Indeed, The Galois representation ρ_π is completely determined, as mentioned in §5.1.5, by information at primes in S_0. If we have sufficiently strong instances of Langlands functoriality available to us then one should be able to recover, from ρ_π, the Hecke information for π at places away from S_0 as well. Note that we aren't saying that π is determined by ρ_π (in fact strong multiplicity one fails for unitary groups) but rather systems of Hecke eigenvalues for π are determined by ρ_π.

Continuing on with the description of an eigenvariety, we make one more choice. We let $e \in \mathcal{C}_c(G(\mathbf{A}_f^{S_0 \cup p}), \overline{\mathbf{Q}})$ be an idempotent. Assume that we have extended it to $\mathcal{C}_c(G(\mathbf{A}_f^p), \overline{\mathbf{Q}}))$ trivially at primes in S_0. The element e acts on automorphic representations π and we let

$$\mathcal{Z}_e = \{(\psi_{\pi,\chi}, \mathbf{k}_\pi) \colon e(\pi^p) \neq (0)\}.$$

For example, we consider again our tame level K^p and we let e_{K^p} be the characteristic function of $K^p \subset G(\mathbf{A}_f^p)$ (scaled so that it is idempotent). In that case, $e_{K^p}(\pi^p) \neq 0$ if and only if π has tame level dividing K^p. We suppose from now on as well that $ee_{K^p} = e$, i.e. every element of type e has tame level K^p.

We refer to the set $\mathcal{Z}_e$ as the refined automorphic representations of type e. One of the main results of [**15**] (see as well [**5**, Theorem 7.3.1]) is that an eigenvariety for $\mathcal{Z}_e$ exists. If x is a point in a rigid analytic space X we use ev_x to denote the evaluation map $\mathrm{ev}_x : \Gamma(X, \mathcal{O}) \to L(x)$. We also introduce elements $F_i \in \Gamma(X, \mathcal{O})$ given by $F_i(x) = \mathrm{ev}_x \circ \psi(u_i)$, where $u_i \in \mathcal{A}_p$ are given by (5.9).

Proposition 5.3. *An eigenvariety of $\mathcal{Z}_e$ exists. That is, there exists a reduced rigid analytic space X equipped with*

(a) A character $\psi : \mathcal{H} \to \Gamma(X, \mathcal{O})$,
(b) An analytic map $\omega : X \to \mathcal{W}$,
(c) An accumulation subset $Z \subset X(\overline{\mathbf{Q}}_p)$.

such that

(1) The natural map

$$\begin{aligned} X(\overline{\mathbf{Q}}_p) &\to \mathrm{Hom}(\mathcal{H}, \overline{\mathbf{Q}}_p) \times \mathcal{W} \\ x &\mapsto (\mathrm{ev}_x \circ \psi, \omega(x)) \end{aligned}$$

induces a bijection between Z and $\mathcal{Z}_e$.

(2) $\widehat{\mathcal{O}}^{\mathrm{rig}}_{X,x}$ is topologically generated over $\widehat{\mathcal{O}}^{\mathrm{rig}}_{\mathcal{W},\omega(x)}$ by the germs of functions $\psi(H) \subset \mathcal{O}^{\mathrm{rig}}_{X,x}$.

(3) Let $u_0 = \mathrm{diag}(p^{n-1}, \dots, p, 1) \in \mathcal{A}_p^\times$. Then, the map $(\omega, \psi(u_0)^{-1}) : X \to \mathcal{W} \times \mathbf{G}_m$ is finite.

The data (X, ψ, ω, Z) is unique up to unique isomorphism (in an evident sense) from just these three axioms. Moreover, we can also say

(I) X is equidimensional of dimension n.
(II) Let $Z' \subset Z$ be the set of points z such that
(i) $\omega(z) = (k_{1,z}, \dots, k_{n,z}) \in \mathbf{Z}^{n,-}$,
(ii) for each $i = 1, \dots, n-1$ we have

$$v_p\left(F_1(z) \cdots F_i(z)\right) < k_{i,z} - k_{i+1,z} + 1.$$

Then, Z' is Zariski dense in X and accumulates at every point of Z.

Proof. First, the uniqueness statement is [**5**, Lemma 7.2.7]. The existence is given by [**5**, Theorem 7.3.1] except that the theorem as stated only says that Z' accumulates on itself. However, the proof they give clearly implies our statement. We include it for the reader's convenience.

By [**5**, Proposition 7.3.5] we have that any point $z \in X$ satisfying condition (i) and (ii) is necessarily at point of Z, i.e. must be classical. This is an analog in this setting of the classical control theorem due to Coleman [**21**, Theorem 6.1]. To see that these points are dense in an affinoid neighborhood basis of a point of $z \in Z$, consider any open locus $z \in U$ on which $u \mapsto v_p(\mathrm{ev}_u \psi(u_i))$ is constant for each i. Then U is admissibly covered by open affinoids V so that $\omega(V)$ is open $\mathcal{W}$ (see [**5**, Theorem 7.3.1]) and $\omega\big|_V$ is finite (and each F_i has constant slope, still). Recall that any set of the form

$$\mathbf{Z}^{n,-}_c = \left\{\mathbf{k} \in \mathbf{Z}^{n,-} : k_i - k_{i+1} > c\right\}$$

is Zariski dense in $\omega(V)$ since $\omega(z) \in \mathbf{Z}^n$. In particular, taking

$$c = -1 + \max_i v_p(F_1(z) \cdots F_i(z)),$$

we see that the points satisfying (i) and (ii) are dense in V and we conclude this final point. □

5.2. Galois representations over eigenvarieties

The rest of the relevant properties of eigenvarieties, for us, will be described in terms of the family of Galois representations. Recall that Proposition 5.2 says that to a regular algebraic automorphic representation π for G there is associated a Galois representation $\rho_\pi : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying some amount of compatibility with the local Langlands correspondence at a density one set of places. By §5.1.6, if z is a point corresponding to π on an eigenvariety X then ρ_π is completely determined by the unramified part $\mathrm{ev}_z \circ \psi_{\mathrm{unr}} : \mathcal{H}_{S_0}^{\mathrm{unr}} \to L(z)$. In fact, the assignment $\pi \mapsto \rho_\pi$ interpolates along an eigenvariety by the following result.

PROPOSITION 5.4 ([**5**, Proposition 7.5.4]). *For every point $x \in X(\overline{\mathbf{Q}}_p)$ there exists a unique, continuous, semi-simple Galois representation*

$$\rho_x : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

characterized by: for all $w \mid \ell$ with $\ell \in S_0$ we have

$$\det(X - \rho_x(\mathrm{Frob}_w)) = \sum_{i=0}^{n} (-1)^{n-i} \mathrm{ev}_x \circ \psi(x_{i,w}) X^i. \tag{5.12}$$

In particular, $\mathrm{tr}(\rho_x(\mathrm{Frob}_w)) = \mathrm{ev}_x \circ \psi(x_{n-1,w})$.

Here, there x_i are the Satake parameters defined in (5.2) with the subscript w denoting that we are seeing it in the local Hecke algebra via the place w (incuding an isomorphism $\mathcal{H}(G(\mathbf{Q}_\ell), K_\ell) \cong \mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_\ell), \mathrm{GL}_n(\mathbf{Z}_\ell))$). For points $x = z = (\psi_{\pi,\chi}, \mathbf{k}_\pi) \in Z$ associated to an automorphic representation π of regular weight $\mathbf{k}_\pi$, the representation ρ_x is ρ_π (notice it is independent of χ). The calculation of this trace was shown in (5.6).

From the points of Z one can obtain, via the above interpolation, various facts about the Galois representations ρ_x. For example, one immediately deduces from (5.12) that for all $x \in X(\overline{\mathbf{Q}}_p)$ we have that $\rho_x^\perp \cong \rho_x(n-1)$.

Let us make some short remarks on the proof of Proposition 5.4, as we will need it to make refined families of (φ, Γ)-modules as in Chapter 4. What one considers is the pseudocharacter[5]

$$T : G_{E,S} \to \Gamma(Z, \mathcal{O})$$

given by $T(g)(z) = \mathrm{tr}(\rho_z(g))$, where ρ_z has been defined at points of Z by Proposition 5.2. There are two steps now. First, by a clever argument due to Chenevier [**15**, Proposition 7.1.1] one shows that T extends to a pseudocharacter $T : G_{E,S} \to \Gamma(X, \mathcal{O})$. Second, one specializes back to a given point $x \in X(\overline{\mathbf{Q}}_p)$ and constructs a Galois representation by Taylor's theorem [**61**, Theorem 1(2)]. In any case, over the space X we have a pseudocharacter $T : G_{E,S} \to \Gamma(X, \mathcal{O})$ and we make free use of it from here on out.

LEMMA 5.5. *Suppose that $x \in X(\overline{\mathbf{Q}}_p)$ such that ρ_x is absolutely irreducible. Then, there exists an affinoid $x \in U \subset X$ and a representation $\rho : G_{E,S} \to \mathrm{GL}_n(\Gamma(U, \mathcal{O}))$ such that $\mathrm{tr}(\rho) = T\big|_U$ and $\rho \otimes_{\Gamma(U,\mathcal{O})} L(u) = \rho_u$ for all $u \in U$.*

PROOF. Let A be the rigid local ring $A = \mathcal{O}_{X,x}^{\mathrm{rig}}$. By Newton's method, A is a local Henselian ring. It then follows from the absolutely irreducible hypothesis on ρ_x that ρ_x lifts uniquely to a representation $\widetilde{\rho}_x : G_{E,S} \to \mathrm{GL}_n(A)$ (this was proven, independently, by Rouquier [**55**, Corollarie

[5] Recall, if G is a topological group and R is a topological ring then a d-dimensional pseudocharacter is a continuous function $T : G \to R$ satisfying $T(1) = d$, $T(gh) = T(hg)$ and the so-called Frobenius identity relating $T(g_1 g_2 \cdots g_n)$ to the values of T on various subproducts. An example of such a function is $T = \mathrm{tr}\,\rho$ for a continuous representation $\rho : G \to \mathrm{GL}_n(R)$. For a precise definition along with the basic results in the theory of pseudocharacters, please see [**55**]. They also go under the pseudonym pseudorepresentation.

5.2] and Nyssen [**51**, Théorème 1]). By [**5**, Lemma 4.3.7] (please put, for their notations, $M = \widetilde{\rho}_x$) there exists a representation ρ satisfying all but the final condition. The final condition follows by Taylor's theorem if we semi-simplify the representation $\rho \otimes_{\Gamma(U,\mathcal{O})} L(u)$. However, since the irreducibility locus of ρ is Zariski open and ρ_x is absolutely irreducible, we can further shrink U to ensure that there is no need to semi-simplify. □

Remark (Joke). For the first time since first semester calculus I made a reasonable citation of Newton's method and Taylor's theorem in the same paragraph.

Remark (Serious). It is expected that the hypotheses of Lemma 5.5 are true at the points in Z though it is only known if $n \leq 3$ (but, see the paragraph preceeding at the end of §5.1.6).

We are going to be interested in the local properties of ρ on decomposition groups $G_{E_w} \subset G_{E,S}$. If $w \notin S$ then the variation is clear: the representations are all unramified and Proposition 5.4 tells us that $\mathrm{tr}\,\rho(\mathrm{Frob}_w)$ varies analytically over X. It remains to describe the behavior at places of S.

5.2.1. The variation of (φ,Γ)-modules at places $v \mid p$. Fix now a point $x = z \in Z$ such that ρ_z is irreducible. By the Lemma 5.5 we can choose an affinoid open $z \in U \subset X$ such that for all $u \in U$, ρ_u arises via specialization from a bona fide representation ρ over U. By Proposition 1.13 we can construct a (φ,Γ)-module $D := D_{\mathrm{rig}}(\rho|_{G_{E_v}})$ which is the p-adic interpolation of the (φ,Γ)-modules $D_u = D_{\mathrm{rig}}(\rho_u|_{G_{E_v}})$.

To fix notation for the next result, consider the characters $\sigma_i : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ given by $z \mapsto z^{1-i}$. We extend each σ_i to a character on $\mathbf{Q}_p^\times$ trivially at p. Recall as well that prior to Proposition 5.3 we defined functions $F_i \in \Gamma(X,\mathcal{O})$.

Lemma 5.6. *The family of (φ,Γ)-modules D forms a refined family over the space U. The extra data is defined as follows.*

(a) If $u \in U$ then $\omega(u) \in \mathcal{W}(L(u))$ is a character $\left(\mathbf{Z}_p^\times\right)^{\oplus n} \to L(u)^\times$ which we see as a n-tuple of characters $(\omega_1(u),\dots,\omega_n(u))$. The characters δ_i is are defined by

$$\delta_{i,u} := \sigma_i \cdot \omega_i(u)\,\mathrm{unr}(F_i(u)).$$

(b) The set of classical points is defined by Z.

Proof. We need to check that the data (U, D, δ_i, Z) satisfies the axioms (RF1)-(RF4). Let $z \in Z$ be associated to an automorphic representation π with regular weight $\mathbf{k}_\pi = (k_1 > k_2 > \cdots > k_n)$. Recall that $\omega(z) = \delta_{\mathbf{k}}$ is defined in (5.8). From that definition, we have

$$\mathrm{wt}(\delta_{i,z}) = \mathrm{wt}(\omega_i(z)) + \mathrm{wt}(\sigma_i(z)) = -k_i + i - 1$$

By Proposition 5.2 we have that (RF1) is true at points $z \in Z$ as well as the first half of (RF3) and all of (RF2). The second half of (RF3) follows from the computations we made in §5.1.6 because the list of values (5.11) is the same (by definition) as the list of values $(p^{\mathrm{wt}(\delta_{1,z})}F_1(z),\dots,p^{\mathrm{wt}(\delta_{n,z})}F_n(z))$. Define[6] now $\kappa_i \in \Gamma(X,\mathcal{O})$ by $\kappa_i(u) := \mathrm{wt}(\delta_{i,u})$. The rest of (RF1) (that $\mathrm{wt}(\delta_{i,u})$ is a Hodge-Tate-Sen weight of D_u) follows from [**5**, Lemma 7.5.12].

It remains to show the density statements (RF4) holds. We have to consider the points in Z which are non-critical. Let $z \in Z$. Then, as we have already mentioned, the crystalline eigenvalues of φ acting on $D_{\mathrm{cris}}(D_z)$ are given by

$$(\phi_1(z),\dots,\phi_n(z)) = (p^{\kappa_1(z)}F_1(z),\dots,p^{\kappa_n(z)}F_n(z)).$$

[6]The κ_i are formally the ith coordinates of the composition $X \to \mathbf{A}^n \to \mathbf{A}^n$ given by $\log_p \circ \omega$ followed by the affine change of coordinates $(y_1,\dots,y_n) \mapsto (-y_1, 1-y_2,\dots,n-1-y_n)$.

Consider the set Z' defined in Proposition 5.3. Suppose that $z' \in Z'$. Then, we have

$$v_p(\phi_1(z')) = \kappa_1(z') + v_p(F_1(z')) < \kappa_2(z'),$$

and if $i > 1$ then

$$\begin{aligned} v_p(\phi_1(z') \cdots \phi_i(z')) &= \kappa_1(z') + \cdots + \kappa_i(z') + v_p(F_1(z') \cdots F_i(z')) \\ &< \kappa_1(z') + \cdots + \kappa_{i-1}(z') + \kappa_{i+1}(z'). \end{aligned}$$

It follows from Example 2.25 that any point $z' \in Z'$ is non-critical. The axiom (RF4) follows as in the proof of Proposition 5.3. □

5.2.2. The Weil-Deligne representations at places of S away from p. Our goal here is to recall some notation used in [**5**] in order to explain the behavior of the representations $\{\rho_x\}_{x\in X}$ at primes $w \nmid p$ in S.

Let m be an integer and if A is a commutative ring denote by J_m the nilpotent operator on A^m with $J_m(e_i) = e_{i-1}$ for $i = 2, \dots, m$ and $J_m(e_1) = 0$. Recall that if N is a nilpotent matrix over a field k then it has a Jordan form (after passing to $\overline{k}$)

$$N \sim J_{t_1(N)} \oplus \cdots \oplus J_{t_s(N)}$$

where $t_1(N) \geq \cdots \geq t_s(N)$ are integers, uniquely determined by this ordering.

Definition. *If N and N' are two different nilpotent matrices then we say that $N \prec N'$ if*

$$t_1(N) + \cdots + t_i(N) \leq t_1(N') + \cdots + t_i(N'), \qquad \text{for all } i.$$

We say that $N \sim N'$ if $N \prec N'$ and $N' \prec N$.

If N and N' are the same size, over the same field, then this is equivalent to N being in the Zariski closure of the conjugacy class of N'. We don't in general require N and N' to have the same size or the same coefficients.

Example 5.7. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = J_1 \oplus J_1 \prec J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let F be a local field. Suppose that (r, N) is a Weil-Deligne representation of F. Then, if τ is a finite-dimensional irreducible representation of I_F we can consider the τ-isotypic component $r[\tau]$ of r. Recall that for all $g \in W_F$ we have $r(g^{-1})Nr(g) = \|g\| N$. In particular, I_F commutes with N and so N preserves $r[\tau]$. We denote by N_τ the induced nilpotent operator in $\mathrm{End}_{\mathbf{C}}(r[\tau])$.

Definition. *Let (r, N) and (r', N'). We say that $N \prec_{I_F} N'$ if $N_\tau \prec N'_\tau$ for all τ. We say that $N \sim_{I_F} N'$ if $N_\tau \sim N'_\tau$ for all τ.*

Notice that if $N \sim_{I_F} N'$ then $r\big|_{I_F} \cong r'\big|_{I_F}$. Indeed, $N \sim_{I_F} N'$ implies that $\dim r[\tau] = \dim r'[\tau]$ for each finite-dimensional irreducible complex representation τ of I_F.

Example 5.8. Suppose that r is n-dimensional and that $(r, N) \prec_{I_F} (\mathbf{1}^{\oplus n}, 0)$. If $\tau \neq \mathbf{1}$ then $0_\tau = 0$. Since $0_{\mathbf{1}} = J_1^{\oplus n}$ we get

$$\dim r[\tau] = \begin{cases} 0 & \text{if } \tau \neq \mathbf{1}, \\ n & \text{if } \tau = \mathbf{1}, \end{cases}$$

and we see that (r, N) must be unramified.

We put ourselves back into the eigenvariety setting as before. Choose a point $x \in X(\overline{\mathbf{Q}}_p)$ such that ρ_x is irreducible. If the reader wishes, they may just chose $x = z \in Z$. By Lemma 5.5 we know that on some affinoid neighborhood U of x, the representations $\{\rho_u\}_{u\in U(\overline{\mathbf{Q}}_p)}$ all arise from a single representation $\rho_U : G_{E,S} \to \mathrm{GL}_n(\Gamma(U, \mathcal{O}))$. To simplify notation assume that $U = X$. In that case the local ring $\mathcal{O}^{\mathrm{rig}}_{X,u}$ is reduced and thus we can form the total ring of fractions $F_u := \mathrm{Frac}(\mathcal{O}^{\mathrm{rig}}_{X,u})$. This is a product of fields

$$F_u = \prod_{Z(u)} F_{Z(u),u}$$

running over irreducible components $Z(u)$ passing through u. For each $u \in U$ there are then three representations $\rho_u, \widetilde{\rho}_u$ and $\rho^{\mathrm{gen}}_{Z(u)}$ labeled as:

$$\begin{array}{ccc} & & \mathrm{GL}_n(L(u)) \\ & \overset{\rho_u}{\nearrow} & \uparrow \\ G_{E,S} & \xrightarrow{\widetilde{\rho}_u} & \mathrm{GL}_n(\mathcal{O}^{\mathrm{rig}}_{X,u}) \\ & \underset{\rho^{\mathrm{gen}}_{Z(u)}}{\searrow} & \downarrow \\ & & \mathrm{GL}_n(F_{Z(u),u}) \end{array} \tag{5.13}$$

If w is a place of E then for any ρ we can consider $\rho_w := \rho\big|_{G_{E_w}}$, a p-adic representation of the local field E_w. Fix a $w \nmid p\infty$. Going back to the picture (5.13), it is well-known that attached[7] to $\rho_{u,w}$ there is a Weil-Deligne representation $(r_{u,w}, N_{u,w})$. Slightly less well-known is that the same is true for $\rho^{\mathrm{gen}}_{Z(u)}$ (see [**5**, §7.8.4]) and we denote the associated representation by $(r^{\mathrm{gen}}_{Z(u),w}, N^{\mathrm{gen}}_{Z(u),w})$. The decoration is justified because by [**5**, Proposition 7.8.19] we have that $N^{\mathrm{gen}}_{Z(u),w}$ depends only on $Z(u)$ up to $\sim_{I_{E_w}}$. The content of our discussion is contained in the following proposition.

PROPOSITION 5.9. *If $x \in X$ such that for all $z \in Z$ we have $N_{z,w} \prec_{I_{E_w}} N_{x,w}$ then*

$$\widetilde{\rho}_x\big|_{I_{E_w}} \cong \rho_x\big|_{I_{E_w}} \otimes_{L(x)} \mathcal{O}^{\mathrm{rig}}_{X,x}.$$

PROOF. This is given as [**5**, Corollary 7.5.10] but we include the steps leading to the corollary (with reference) for the convenience of the reader. The first step is that for each irreducible component $Z(x)$ passing through x, there exists a point $z \in Z$ such that $N^{\mathrm{gen}}_{Z(x)} \sim_{I_{E_w}} N_z$. Indeed, in general we have that $N_u \prec_{I_{E_w}} N^{\mathrm{gen}}_{Z(u)}$ for any u ([**5**, Proposition 7.8.19(iii)]) with equality for a Zariski open and dense set of points on a fixed component $Z(u)$ (by [**5**, Proposition 7.8.19(ii)]). Taking $u = x$ and using that Z is Zariski dense we get the first claim.

It suffices, by [**5**, Proposition 7.8.9], to show that for each irreducible component $Z(x)$ we have that $N^{\mathrm{gen}}_{Z(x)} \sim_{I_{E_w}} N_x$. However, the assumption that $N_{z,w} \prec_{I_{E_w}} N_{x,w}$ for all $z \in Z$ implies that $N^{\mathrm{gen}}_{Z(x)} \prec_{I_{E_w}} N_x$ by the first claim. As we already mentioned, though, we have $N_x \prec_{I_{E_w}} N^{\mathrm{gen}}_{Z(x)}$ and thus we get our claim. □

---

[7] If $r : W_{E_w} \to \mathrm{GL}_n(B)$ (for any ring B) is any representation then we say that r admits a Weil-Deligne representation if there exists a nilpotent operator $N \in M_d(B)$ such that

$$r(g) = \exp(t_w(g)N)$$

for all g in an open subgroup of I_{E_w}. Here $t_w : I_{E_w} \to \mathbf{Q}_p$ is any continuous group homomorphism and exp is the usual exponential, well-defined as N is nilpotent and commutes with $t_w(I_{E_w})$. If r admits the structure of a Weil-Deligne representation then the choice of t_w is irrelevant and N is unique.

5.3. The local geometry of eigenvarieties via deformations of Galois representations

We are now ready to study the local geometry of an eigenvariety via the the deformation theory of Galois representations. Below we describe two representable deformation problems[8] $\mathfrak{X}^{\mathrm{PUNK}} \subset \mathfrak{X}^{\mathrm{PK}}$ of Galois representations. We show that the completed local ring of any idempotent eigenvariety at a classical point is naturally a quotient the universal deformation ring for $\mathfrak{X}^{\mathrm{PK}}$ and if we further restrict our eigenvariety (to what we call monodromic) we will see the local ring as a quotient of the smaller problem $\mathfrak{X}^{\mathrm{PUNK}}$. Under some extra hypotheses we also deduce smoothness results of eigenvarieties at classical points.

Throughout this section we will fix an automorphic representation π such that:

(a) π_∞ has regular weight,
(b) the representation $\rho_\pi : G_{E,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ defined in Proposition 5.2 is irreducible (take the same set S as there), and
(c) π_p is unramified and thus ρ_π is crystalline at places $v \mid p$.

Fix the place $v \mid p$. We assume as well

(d) for $w \in S$ such that w is non-split in E, the nilpotent monodromy operator $N_w(\rho_\pi)$ is zero,
(e) we have chosen an accessible refinement χ of the smooth representation π_v of $\mathrm{GL}_n(\mathbf{Q}_p)$ deduced from the isomorphism $G(\mathbf{Q}_p) \cong_v \mathrm{GL}_n(\mathbf{Q}_p)$, and
(f) the representation $\rho_\pi\big|_{G_{E_v}}$ is φ-regular with respect to the refinement χ (see the definition on page 74): this corresponds to the condition that if $\chi = \chi_1 \cdots \chi_n$ then $\chi_i(p)\chi_j(p)^{-1} \notin \{1, p\}$ for all $i < j$ as well as asking that for all i the number $\chi_1(p)\cdots\chi_i(p)$ appears once in the list $\{\chi_{j_1}(p)\cdots\chi_{j_a}(p) \colon j_1 < j_2 < \cdots < j_a\}$ (see the discussion at the end of §5.1.6).

Then, given the data of just (a) and (e) we have, for any idempotent e such that $e(\pi^p) \neq 0$ an eigenvariety X of type e. We fix X now, but remember that implicit is the choice of the idempotent. The pair (π, χ) of the representation together with its refinement defines a point $z \in X(\overline{\mathbf{Q}}_p)$.

5.3.1. The deformation problems. First, we begin with the formal deformation functor $\mathfrak{X}_{\rho_z} : \mathfrak{AR}_{L(z)} \to \underline{\mathrm{Set}}$ given by

$$\mathfrak{X}_{\rho_z}(A) := \{\text{deformations } \rho_A \text{ of } \rho_z \text{ to } A\}_{/\sim}.$$

Since we have assumed that ρ_z is absolutely irreducible, $\mathfrak{X}_{\rho_z}$ is representable by a complete local noetherian ring $R^{\mathrm{univ}}_{\rho_z}$. We denote by ρ_z^{univ} the universal representation.

Let $D_z = D_{\mathrm{rig}}(\rho_{z,v})$ be the (φ,Γ)-module corresponding to the p-adic representation $\rho_{z,v} := \rho_z\big|_{G_{E_v}} : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(L(z))$. Recall that the formal deformation functors $\mathfrak{X}_{D_z}$ and $\mathfrak{X}_{\rho_{z,v}}$ are canonically isomorphic by Lemma 3.2. The choice of the refinement χ at the point z defines a triangulation P_z of D_z. Indeed, the refinement defines an ordering of the crystalline eigenvalues by §5.1.6, which in turn determines a unique triangulation of D_z because z is φ-regular. Recall that in §3.2.4 we defined a subfunctor $\mathfrak{X}^{\mathrm{par},\wedge}_{\rho_{z,v},P_z} \subset \mathfrak{X}_{\rho_{z,v}}$ which parameterized deformations which were

- paraboline with respect to the maximal non-critical parabolization P_z^{nc} of P_z, and
- of Kisin-type on each associated graded of P_z^{nc}.

Since we have assumed that $\rho_{z,v}$ is φ-regular we know by Lemma 3.37 that $X^{\mathrm{par},\wedge}_{\rho_{z,v},P_z}$ is relatively representable. With this in mind we define

$$\mathfrak{X}^{\mathrm{PK}}_{\rho_z} := \mathfrak{X}_{\rho_z} \times_{\mathfrak{X}_{\rho_{z,v}}} \mathfrak{X}^{\mathrm{par},\wedge}_{\rho_{z,v},P_z}.$$

[8]For the reader who is interested: the superscript PK refers to **P**araboline and **K**isin-type and PUNK refers to "**P**araboline, **U**nitary, mo**N**odromy and **K**isin-type. Actually, the last bit is a lie—the N is just the typical symbol for mondoromy and PUMK isn't a word.

By the relative representability of $\mathfrak{X}^{\mathrm{par},\wedge}_{\rho_{z,v},P_z} \to \mathfrak{X}_{\rho_z,v}$ and Proposition 3.1 we have that $\mathfrak{X}^{\mathrm{PK}}_{\rho_z}$ is representable by a quotient $R^{\mathrm{PK}}_{\rho_z}$ of $R^{\mathrm{univ}}_{\rho_z}$. Denote the universal deformation by ρ_z^{PK}. Though this only takes into account the deformation theory at the place v, it already cuts out quite and interesting ring.

Proposition 5.10. *There is a natural surjection* $R^{\mathrm{PK}}_{\rho_z} \twoheadrightarrow \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$.

Proof. By Lemma 5.5 we can lift the representation ρ_z to a representation $\rho : G_{E,S} \to \mathrm{GL}_n(\Gamma(U,\mathcal{O}))$ over an affinoid neighborhood U of z. Furthermore, Lemma 5.6 implies that the (φ,Γ)-module $D := D_{\mathrm{rig}}(\rho\big|_{G_{E_v}})$ forms a refined family of (φ,Γ)-modules over U. We freely use the notation (e.g. the δ_j and F_i) of that lemma as well defining, for $a < b$, the character

$$\Delta_{a,b} := \delta_a \cdots \delta_b.$$

Since we assume that $\rho_{z,v}$ is φ-regular we know, by Theorem 4.13, that after shrinking U we can assume that D has a parabolization P deforming the maximal non-critical paraboliztion P_z^{nc}. To fix notation, we let

$$P : 0 = P_{i_0} \subsetneq P_{i_1} \subsetneq \cdots \subsetneq P_{i_s} = D.$$

Denote as well $\mathrm{Gr}_k\, P = P_{i_k}/P_{i_{k-1}}$. This is also a refined family of (φ,Γ)-modules with respect to the same set of classical points (or at least that many) and the characters $\delta_{i_{k-1}+1},\ldots,\delta_{i_k}$. Moreover, the point z is still φ-regular with respect to this family. Thus, after shrinking U even more, we can assume, by Lemma 4.16, that the cohomology groups $H^0\left((\wedge^j \mathrm{Gr}_k\, P)\,(\Delta^{-1}_{i_{k-1}+1,j})\right)$ (for any possible k and j) are free of rank one and satisfy base change. Over U then we have embeddings

$$\mathcal{R}_X(\mathrm{unr}(F_{i_{k-1}}\cdots F_j)) \hookrightarrow \left(\wedge^j \mathrm{Gr}_k\, P\right)(\Delta_{i_{k-1}+1,j}\big|_{\Gamma}^{-1}).$$

Notice that since the quotient is projective over the non-critical locus, the base change of this map is injective at any point.

Now suppose that I is a co-finite length ideal inside $\mathcal{O}^{\mathrm{rig}}_{X,z}$. From what we have said above, it follows that:

- D/ID is a paraboline deformation of D_z with respect to P_z^{nc} over A/I, and
- $D_{\mathrm{cris}}(\wedge^j \mathrm{Gr}_k\, P/IP(\Delta_{i_{k-1}+1,j}\big|_{\Gamma})^{-1})^{\varphi=F_{i_{k-1}+1}\cdots F_j}$ is free of rank one (by φ-regularity) over A/I for each $j = 1,\ldots,\mathrm{rank}\,\mathrm{Gr}_k\, P$ and for each $k = 1,\ldots,s$.

Thus D/ID defines a point in $\mathfrak{X}^{\mathrm{par},\wedge}_{\rho_{z,v},P_z}(\mathcal{O}^{\mathrm{rig}}_{X,z}/I)$ for each I. By definition this implies that $\rho \otimes_X \mathcal{O}^{\mathrm{rig}}_{X,z}/I$ defines a point of $\mathfrak{X}^{\mathrm{PK}}_{\rho_z}(\mathcal{O}^{\mathrm{rig}}_{X,z}/I)$. Taking I ranging over all possibilities we deduce that there is a unique morphism $\vartheta : R^{\mathrm{PK}}_{\rho_z} \to \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ such that $\vartheta \circ \rho_z^{\mathrm{PK}} = \rho \otimes_X \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$.

The argument that the map is surjective is standard, but we include it for convenience. Since $R^{\mathrm{PK}}_{\rho_z}$ is compact, it suffices by Proposition 5.3 to show $\widehat{\mathcal{O}}^{\mathrm{rig}}_{\mathcal{W},\omega(z)}[\psi(\mathcal{H})]$ is contained in $\vartheta(R^{\mathrm{PK}}_{\rho_z})$. The weight space $\mathcal{W}$ is a disjoint union of polydiscs and we have explicit coordinates

$$\widehat{\mathcal{O}}^{\mathrm{rig}}_{\mathcal{W},\omega(z)} \cong \mathbf{Q}_p[\![\log_p \omega_1,\ldots,\log_p \omega_n]\!].$$

Then, if $\kappa_1^{\mathrm{PK}},\ldots,\kappa_n^{\mathrm{PK}} \in R^{\mathrm{PK}}_{\rho_z}$ are the Hodge-Tate-Sen weights of ρ_z^{PK} we have, by Lemma 5.6, that $\vartheta(\kappa_i^{\mathrm{PK}}) = \log_p \omega_i + 1 - i$. Thus, $\widehat{\mathcal{O}}^{\mathrm{rig}}_{\mathcal{W},\omega(z)} \subset \vartheta(R^{\mathrm{PK}}_{\rho_z})$. For the Hecke part of $\widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ we are charged with

(a) showing that for $w \in S_0$ such that π_ℓ is unramified, the elements $\psi(x_{i,w})$ are in the image of ϑ, and
(b) $\vartheta(R^{\mathrm{PK}}_{\rho_\pi})$ also contains the elements $\psi(u_{i,v}) = F_i$ (for $i = 0,1,\ldots,n-1$).

If $w \in S_0$ and π_ℓ is unramified, then (5.12) shows that all the elements of (a) are in the image of ϑ, arising as taking ϑ of the characteristic polynomial of $\rho_z^{\mathrm{PK}}(\mathrm{Frob}_w)$ (which has $R_{\rho_z}^{\mathrm{PK}}$-coefficients). We finally have, by definition, that there are elements $F_1^{\mathrm{PK}}, \dots, F_n^{\mathrm{PK}}$ in $R_{\rho_z}^{\mathrm{PK}}$ such that for any co-finite length ideal $J \subset R_{\rho_z}^{\mathrm{PK}}$

$$D_{\mathrm{cris}}^{+}\left(\wedge^i\left(\rho_z^{\mathrm{PK}}\big|_{G_{E_v}}/J\right)(\kappa_1^{\mathrm{PK}} + \cdots \kappa_i^{\mathrm{PK}})\right)^{\varphi = F_1^{\mathrm{PK}} \cdots F_i^{\mathrm{PK}}}$$

is free of rank one over $R_{\rho_\pi}^{\mathrm{PK}}/J$. Taking $i = 1, 2, \dots, n$ we have that $\vartheta(F_i^{\mathrm{PK}}) = F_i$. □

We continue now to further impose deformation conditions at places $w \in S$ with $w \nmid p$. To that end, we consider a place $w \in S$ with $w \nmid p$. We then have the subfunctor $\mathfrak{X}_{\rho_{z,w},f} \subset \mathfrak{X}_{\rho_{z,w}}$ defined by the unramified Bloch-Kato condition. Specifically, $\rho_A \in \mathfrak{X}_{\rho_{z,w},f}(A)$ if and only if $\rho_A\big|_{I_{E_w}} \cong \rho_z\big|_{I_{E_w}} \otimes_{L(z)} A$. It is easy to check that the conditions of Proposition 3.4 are satisfied and thus $\mathfrak{X}_{\rho_{z,w},f}$ is a relatively representable subfunctor of $\mathfrak{X}_{\rho_{z,w}}$. Its tangent space is the well-known local unramified Bloch-Kato Selmer group

$$\mathfrak{t}_{\rho_{z,w},f} = H_f^1(G_{E_w}, \operatorname{ad}\rho_{z,w})$$

which parameterizes extensions V of the trivial representation by $\operatorname{ad}\rho_{z,w}$ such that the sequence

$$0 \to (\operatorname{ad}\rho_{z,w})^{I_{E_w}} \to V^{I_{E_w}} \to L(z) \to 0$$

is still exact. We can package this together with the crystalline deformation functors at places diving p and obtain a relatively representable subfunctor $\mathfrak{X}_{\rho_z,f} \subset \mathfrak{X}_{\rho_z}$. Its Zariski tangent space $\mathfrak{t}_{\rho_z,f}$ is the global Bloch-Kato Selmer group

$$\mathfrak{t}_{\rho_z,f} = H_f^1(G_{E,S}, \operatorname{ad}\rho_z) = \ker\left(H^1(G_{E,S}, \operatorname{ad}\rho_z) \to \prod_{w \in S} H^1(G_{E_w}, \operatorname{ad}\rho_z)/H_f^1(G_{E_w}, \operatorname{ad}\rho_z)\right). \tag{5.14}$$

Continuing on, we also have a subfunctor $\mathfrak{X}_{\rho_z}^{\mathrm{U}} \subset \mathfrak{X}_{\rho_z}$ given on points by:

$$\mathfrak{X}_{\rho_z}^{\mathrm{U}}(A) = \left\{\rho_A \in \mathfrak{X}_{\rho_z}(A) \colon \rho_A^{\perp} \cong \rho_A(n-1)\right\}.$$

As before, the criterion Proposition 3.4 makes it clear that this is relatively representable. We now define a deformation problem $\mathfrak{X}_{\rho_z}^{\mathrm{PUNK}} \subset \mathfrak{X}_{\rho_z}$ by the fibered product

$$\begin{array}{ccc} \mathfrak{X}_{\rho_z}^{\mathrm{PUNK}} & \longrightarrow & \mathfrak{X}_{\rho_z} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\rho_z}^{\mathrm{U}} \times \mathfrak{X}_{\rho_{z,v},P_z}^{\mathrm{par},\wedge} \times \prod_{w \in S^p} \mathfrak{X}_{\rho_{z,w},f} & \longrightarrow & \mathfrak{X}_{\rho_z} \times \mathfrak{X}_{\rho_{z,v}} \times \prod_{w \in S^p} \mathfrak{X}_{\rho_{z,w}}. \end{array} \tag{5.15}$$

Here, S^p denotes the set of places $w \in S$ such that $w \nmid p$. The bottom arrow is the natural inclusion into each coordinate. By what we just explained the subfunctor $\mathfrak{X}_{\rho_z}^{\mathrm{PUNK}}$ is relatively representable. We have the following upper bound on the deformation space cut out by $\mathfrak{X}_{\rho_z}^{\mathrm{PUNK}}$. It relates a deformation space of a global Galois representation to the local deformation space.

LEMMA 5.11. *The deformation functor $\mathfrak{X}_{\rho_z,f}$ is a subfunctor of $\mathfrak{X}_{\rho_z}^{\mathrm{PUNK}}$ and we have an inclusion*

$$\mathfrak{t}_{\rho_z}^{\mathrm{PUNK}}/\mathfrak{t}_{\rho_z,f} \hookrightarrow \mathfrak{t}_{\rho_{z,v},P_z}^{\mathrm{par},\wedge}/\mathfrak{t}_{\rho_{z,v},f}.$$

PROOF. Every deformation in $\mathfrak{t}_{\rho_z}^{\mathrm{PUNK}}$ is unramified in the sense of Bloch-Kato away from p. Notice that we have not explicitly specified a deformation condition at the place v^c. However, if ρ_A is a deformation in $\mathfrak{X}_{\rho_z}^{\mathrm{U}}$ then $\rho_{A,v^c}^{\vee} \cong \rho_{A,v}(n-1)$ and any deformation condition at v implicitly defines a deformation condition at v^c. In particular, a deformation $\rho_A \in \mathfrak{X}^{\mathrm{U}}_{\rho_z}$ is crystalline at v if and only if it is crystalline at v^c. The result then follows. $\square$

5.3.2. Towards an $R = \mathbf{T}$ result. The question remains whether or not $\widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ is naturally[9] cut out by a deformation problem. Notice that the ring $R^{\mathrm{PUNK}}_{\rho_z}$ is naturally a quotient of $R^{\mathrm{PK}}_{\rho_z}$ but that it is not necessary that the map $R^{\mathrm{PK}}_{\rho_z} \to \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ factor through this quotient. In order to make it so we restrict our eigenvariety slightly.

Recall that the definition of X depended on the choice of an idempotent $e \in \mathcal{C}_c(G(\mathbf{A}^p_f), \overline{\mathbf{Q}})$. The choice of our point z and the representation π at the start of §5.3 remains in force. In particular notice the hypothesis on the monodromy operator of ρ_z at places $w \in S$ which are non-split in E.

Definition. *A monodromic eigenvariety X for z is one which arises from an idempotent e such that if $e(\pi') \neq 0$ then $N_w(\rho_{\pi'}) \prec_{I_{E_w}} N_w(\rho_z)$ for all w.*

We should note that monodromic eigenvarieties exist (see the "minimal eigenvarieties" explained in [**5**, Example 7.5.1]). In fact, one should perhaps state everything in automorphic terms but as our study is Galois-theoretic in nature we prefer to state it this way. One of the difficulties that arise when stating it in terms of automorphic representations is that the notion of monodromy at places $w \in S$ diving a place ℓ where π_ℓ is ramified and ℓ is non-split is unclear. In their original work [**5**], Bellaïche and Chenevier dealt with this by studying/defining the notion of non-monodromic principal series representations (see §6.6 of *loc. cit.*). Such representations fall under our assumption on $N_w(\rho_z)$ at such places.

On the other hand, at places $w \in S$ where w lies above a split prime then the local Langlands correspondence provides us, for each π' with a Weil-Deligne representation $(r_w(\pi'), N_w(\pi'))$. The relation we have is that $r_w(\pi') = r_w(\rho_{\pi'})$ and that $N_w(\rho_{\pi'}) = N_w(\pi')$. In particular, if e actually cuts out all the π' such that $N_w(\pi') \prec_{I_{E_w}} N_w(\pi)$ for split w (which is what the minimal eigenvarieties do) then e defines a monodromic eigenvariety. In any case, we now fix a monodromic eigenvariety X for the point z. The upshot of controlling the monodromy action is we can prove the following result.

Theorem 5.12. *Suppose that X is a monodromic eigenvariety for π. Then, the natural map $R^{\mathrm{PK}}_{\rho_z} \to \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ factors through the quotient $R^{\mathrm{PUNK}}_{\rho_z}$.*

Proof. Recall that we denote by $\widehat{\rho}_z$ the deformation of ρ_z to the completed local ring $\widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$. We've already shown that this defines a point of $R^{\mathrm{PK}}_{\rho_z}$ and so we need to show that $\widehat{\rho}_z$ satisfies the other conditions in the definition (5.15) of $\mathfrak{X}^{\mathrm{PUNK}}_{\rho_z}$. However, the conjugate self-dual condition follows from our discussion proceeding Proposition 5.4. The fact that $\widehat{\rho}_z$ is constant on inertia at places $w \in S^p$ follows from Proposition 5.9. $\square$

We immediately use this result to obtain an upper bound on the Zariski tangent space of X at the point z.

Corollary 5.13. *Assume that $H^1_f(G_{E,S}, \mathrm{ad}\,\rho_z) = (0)$. Then,*

$$\dim \mathfrak{t}_{X,z} \leq \dim \mathfrak{t}^{\mathrm{par},\wedge}_{\rho_{z,v},P_z} / \mathfrak{t}_{\rho_{z,v},f}.$$

Proof. The surjection $R^{\mathrm{PUNK}}_{\rho_z} \twoheadrightarrow \widehat{\mathcal{O}}^{\mathrm{rig}}_{X,z}$ defines an inclusion $\mathfrak{t}_{X,z} \hookrightarrow \mathfrak{t}^{\mathrm{PUNK}}_{\rho_z}$. The result then follows by Lemma 5.11, noting the hypothesis. $\square$

[9]It is evidently cut out by *some* subfunctor, the question is whether or not we can describe it

Recall that in Chapter 3, we considered a hypothesis (3.15) on upper bounds for certain Kisin-type deformation functors. In particular, we can consider (3.15) for associated gradeds of P_z^{nc}, each of which is equipped with a fully critical triangulation.

THEOREM 5.14. *Assume that the each associated graded of P_z^{nc} satisfies the hypothesis* (3.15) *of Chapter 3. Then, the natural map $R_{\rho_z}^{\mathrm{PUNK}} \to \widehat{\mathcal{O}}_{X,z}^{\mathrm{rig}}$ is an isomorphism.*

PROOF. We know in general that $\dim \mathfrak{t}_{X,z} \geq n$ because X is equidimensional of dimension n by Proposition 5.3. On the other hand, by Theorem 3.38 we know that $\dim \mathfrak{t}_{\rho_{z,v},P_z}^{\mathrm{par},\wedge}/\mathfrak{t}_{\rho_{z,v},f} \leq n$. The hypothesis there is valid because of our assumption on P_z. Thus, the previous corollary implies that we have

$$n \leq \dim \mathfrak{t}_{X,z} \leq \dim \mathfrak{t}_{\rho_{z,v},P_z}^{\mathrm{par},\wedge}/\mathfrak{t}_{\rho_{z,v},f} \leq n,$$

forcing equality throughout. In particular, we also have that $\dim \mathfrak{t}_{\rho_z}^{\mathrm{PUNK}} = n$ (being squeezed in the middle). General principles (in the deformation theory) then imply that

$$R_{\rho_z}^{\mathrm{PUNK}} \cong L(z)[\![h_1,\ldots,h_n]\!]/I \twoheadrightarrow \widehat{\mathcal{O}}_{X,z}^{\mathrm{rig}} \tag{5.16}$$

for some ideal $I \subset L(z)[\![h_1,\ldots,h_n]\!]$. However, since $\dim \widehat{\mathcal{O}}_{X,z}^{\mathrm{rig}} = n$ we see that $I = (0)$ and the map (5.16) must be an isomorphism. □

We give one example of a smoothness result that can be applied in a vacuum.

COROLLARY 5.15. *Suppose that $n \leq 3$. Suppose as well that $\rho_{z,v}$ is indecomposable and $H_f^1(G_{E,S}, \mathrm{ad}\,\rho_z) = (0)$. Then, any monodromic eigenvariety X containing z is smooth at z.*

PROOF. This follows from Theorem 5.14 and Proposition 3.40. □

We end now with some general remarks.

REMARK. Note that the condition that $\rho_{z,v}$ be indecomposable in Corollary 5.15 is independent of the triangulation P_z at the point $z = (\pi, P_z)$. In particular, for a fixed π it can be applied to *any* refined point $z = (\pi, P_z)$.

REMARK. The hypothesis that $H_f^1(G_{E,S}, \mathrm{ad}\,\rho_z) = (0)$ in each of the above corollaries should be seen as a technical one. Note that the hypothesis can be checked after making a finite base change E'/E since $\mathrm{ad}\,\rho_z$ is a characteristic zero representation. The hypothesis can then be deduced in any situation where we have available a potential automorphy theorem and the corresponding $R = \mathbf{T}$ result. We note as well that it is conjectured to always be true, and known in almost every case if $\dim \rho_z \leq 2$. The reader should see [**5**, Chapter 5] for a general discussion.

REMARK. Consider Corollary 5.15 again. By [**16**, Theorem 4.8] one knows that if z is a non-critical point then the weight map $\omega : X \to \mathcal{W}$ is étale at the point z. Furthermore, the arguments of [**7**] can be adapted to the situation of Corollary 5.15 to show that ω is étale at z if and only if z is non-critical. These arguments, unfortunately, did not make it into this thesis, but they will appear in future work.

REMARK. A previous study by Bellaïche [**2**] of critical points on an eigenvariety for U(3) actually produced non-smooth points, contrary to Corollary 5.15. Those points, however, all arose in situations where the global representation ρ_z was decomposable. It still seems unclear what the precise role the irreducibility of ρ_z plays in smoothness results. The author offers no conjecture.

Bibliography

[1] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor. Local-global compatibility for $l = p$, II. *Ann. Sci. École Norm. Sup. (4)*, 2012. to appear.

[2] J. Bellaïche. Nonsmooth classical points on eigenvarieties. *Duke Math. J.*, 145(1):71–90, 2008.

[3] J. Bellaïche. Critical p-adic L-functions. *Invent. Math.*, 189(1):1–60, 2012.

[4] J. Bellaïche and G. Chenevier. Lissité de la courbe de Hecke de GL_2 aux points Eisenstein critiques. *J. Inst. Math. Jussieu*, 5(2):333–349, 2006.

[5] J. Bellaïche and G. Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[6] D. Benois. A generalization of Greenberg's $\mathscr{L}$-invariant. *Amer. J. Math.*, 133(6):1573–1632, 2011.

[7] J. Bergdall. Ordinary modular forms and companion points on the eigencurve. *Preprint*, 2012. Submitted, available at `arXiv:1210.0165`.

[8] L. Berger. Représentations p-adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.

[9] L. Berger. Équations différentielles p-adiques et (ϕ, N)-modules filtrés. *Astérisque*, (319):13–38, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[10] S. Bloch and K. Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[11] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[12] K. Buzzard. Eigenvarieties. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 59–120. Cambridge Univ. Press, Cambridge, 2007.

[13] A. Caraiani. Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.*, 161(12):2311–2413, 2012.

[14] A. Caraiani. Monodromy and local-global compatibility for $l = p$. *preprint*, 2012.

[15] G. Chenevier. Familles p-adiques de formes automorphes pour GL_n. *J. Reine Angew. Math.*, 570:143–217, 2004.

[16] G. Chenevier. On the infinite fern of Galois representations of unitary type. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(6):963–1019, 2011.

[17] G. Chenevier. Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. *Math. Ann.*, 355(4):1469–1525, 2013.

[18] G. Chenevier and M. Harris. Construction of automorphic galois reprsentations, II. *Cambridge Journal of Mathematics*, 1(1), 2013.

[19] F. Cherbonnier and P. Colmez. Représentations p-adiques surconvergentes. *Invent. Math.*, 133(3):581–611, 1998.

[20] R. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[21] R. F. Coleman. Classical and overconvergent modular forms. *Invent. Math.*, 124(1-3):215–241, 1996.

[22] R. F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

[23] P. Colmez. Représentations triangulines de dimension 2. *Astérisque*, (319):213–258, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[24] P. Deligne. Formes modulaires et représentations l-adiques. *Séminaire N. Bourbaki*, (355):139–172, 1968-1969.

[25] M. Eichler. Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. *Arch. Math.*, 5:355–366, 1954.

[26] M. Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.

[27] J.-M. Fontaine. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

[28] J.-M. Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).
[29] J.-M. Fontaine. Représentations p-adiques semi-stables. *Astérisque*, (223):113–184, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).
[30] J.-M. Fontaine and B. Mazur. Geometric Galois representations. In *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Ser. Number Theory, I, pages 41–78. Int. Press, Cambridge, MA, 1995.
[31] F. Gouvêa and B. Mazur. Families of modular eigenforms. *Math. Comp.*, 58(198):793–805, 1992.
[32] F. Q. Gouvêa. *Arithmetic of p-adic modular forms*, volume 1304 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
[33] A. Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. II. Le théorème d'existence en théorie formelle des modules. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 195, 369–390. Soc. Math. France, Paris, 1995.
[34] M. Harris and J.-P. Labesse. Conditional base change for unitary groups. *Asian J. Math.*, 8(4):653–683, 2004.
[35] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.
[36] L. Herr. Sur la cohomologie galoisienne des corps p-adiques. *Bull. Soc. Math. France*, 126(4):563–600, 1998.
[37] H. Hida. Congruence of cusp forms and special values of their zeta functions. *Invent. Math.*, 63(2):225–261, 1981.
[38] H. Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.
[39] N. M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973.
[40] K. Kedlaya and R. Liu. On families of ϕ, Γ-modules. *Algebra Number Theory*, 4(7):943–967, 2010.
[41] K. Kedlaya, J. Pottharst, and L. Xiao. Cohomology of arithmetic families of (φ, γ)-modules. *preprint*, 2012.
[42] K. S. Kedlaya. A p-adic local monodromy theorem. *Ann. of Math. (2)*, 160(1):93–184, 2004.
[43] M. Kisin. Overconvergent modular forms and the Fontaine-Mazur conjecture. *Invent. Math.*, 153(2):373–454, 2003.
[44] J.-P. Labesse. Changement de base CM et séries discrètes. In *On the stabilization of the trace formula*, volume 1 of *Stab. Trace Formula Shimura Var. Arith. Appl.*, pages 429–470. Int. Press, Somerville, MA, 2011.
[45] M. Lazard. Les zéros des fonctions analytiques d'une variable sur un corps valué complet. *Inst. Hautes Études Sci. Publ. Math.*, (14):47–75, 1962.
[46] R. Liu. Cohomology and duality for (ϕ, Γ)-modules over the Robba ring. *Int. Math. Res. Not. IMRN*, (3):Art. ID rnm150, 32, 2008.
[47] D. Loeffler. Overconvergent algebraic automorphic forms. *Proc. Lond. Math. Soc. (3)*, 102(2):193–228, 2011.
[48] B. Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.
[49] B. Mazur. An introduction to the deformation theory of Galois representations. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.
[50] B. Mazur and A. Wiles. On p-adic analytic families of Galois representations. *Compositio Math.*, 59(2):231–264, 1986.
[51] L. Nyssen. Pseudo-représentations. *Math. Ann.*, 306(2):257–283, 1996.
[52] S. Patrikis and R. Taylor. Automorphy and irreducibility of some l-adic representations. *preprint*, 2013. Available at `http://www.math.ias.edu/~rtaylor/`.
[53] J. Pottharst. Analytic families of finite-slope selmer groups. *preprint*, 2011.
[54] J. D. Rogawski. *Automorphic representations of unitary groups in three variables*, volume 123 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.
[55] R. Rouquier. Caractérisation des caractères et pseudo-caractères. *J. Algebra*, 180(2):571–586, 1996.
[56] T. Saito. Modular forms and p-adic Hodge theory. *Invent. Math.*, 129(3):607–620, 1997.
[57] S. Sen. Continuous cohomology and p-adic Galois representations. *Invent. Math.*, 62(1):89–116, 1980/81.
[58] J.-P. Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972)*, pages 191–268. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.
[59] G. Shimura. Correspondences modulaires et les fonction de courbes algébrique. *J. Math. Soc. Japan*, (10):28, 1958.
[60] S. W. Shin. Galois representations arising from some compact Shimura varieties. *Ann. of Math. (2)*, 173(3):1645–1741, 2011.

[61] R. Taylor. Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.*, 63(2):281–332, 1991.

[62] R. Taylor and A. Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[63] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[64] A. Wiles. On p-adic representations for totally real fields. *Ann. of Math. (2)*, 123(3):407–456, 1986.

[65] A. Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.