**On the distribution of modular points on deformation rings**

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(joint work with Chengyang Bao, Brandon Levin)

Let $p$ be an odd prime. Fix an odd and continuous Galois representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p).$$

Serre's conjecture implies $\overline{\rho}$ has *modular lifts* to level $N(\overline{\rho}) = N$, which are the Galois representations

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

attached to eigenforms $f \in S_k(\Gamma_1(N))$ such that $\overline{\rho}_f \simeq \overline{\rho}$. Let $D$ be a decomposition group at $p$, and assume $\overline{r} = \overline{\rho}|_D$ has only scalar endomorphisms. Mazur defined ([9]) the universal deformation space $X_{\overline{r}}$. For $k \geq 2$ we consider the natural closed subscheme of crystalline deformations $X_{\overline{r}}^k \subseteq X_{\overline{r}}$, defined by Kisin ([5]), whose generic fiber parametrizes crystalline lifts of $\overline{r}$ that have Hodge–Tate weights $0 < k-1$ and cyclotomic determinant. Since $N$ is prime to $p$, the restrictions $\rho_f|_D$ of modular lifts are crystalline and hence define *modular points* on $X_{\overline{r}}^k$ as $k$ varies. Kisin showed ([6]), in landmark modularity results for $\mathrm{GL}_{2/\mathbf{Q}}$, each irreducible component of $X_{\overline{r}}^k$ supports a modular point. We aim to answer two questions:

(1) How many modular points are on a fixed component in $X_{\overline{r}}^k$?
(2) How many components does $X_{\overline{r}}^k$ have?

To answer Question 1, recall that Serre weights are the irreducible $\mathbf{F}_p$-linear $\mathrm{GL}_2(\mathbf{F}_p)$-representations $\sigma_{a,b} = \mathrm{Sym}^a(\det^b)$ with $a \leq p-1$. They enter the story in two ways. First, assuming $\sigma = \sigma_{a,b}$, we may define

$$N_{\mathrm{aut}}(\sigma) = \dim S_{a+2}(\Gamma_1(N))[\overline{\rho} \otimes \chi_{\mathrm{cyc}}^{-b}].$$

For context, the weight part of Serre's conjecture says that if $N_{\mathrm{aut}}(\sigma) \neq 0$, then $\sigma$ is a Serre weight for $\overline{\rho}$. Second, writing $\overline{X}_{\overline{r}}^k \subseteq \overline{X}_{\overline{r}}$ for the special fibers, there are canonical one-dimensional *Breuil–Mézard cycles* $C(\sigma)$ in $\overline{X}_{\overline{r}}$. The geometric Breuil–Mézard conjecture of Emerton and Gee ([4]) implies the cycle of $\overline{X}_{\overline{r}}^k$ is

$$[\overline{X}_{\overline{r}}^k] = \sum_{\sigma} \begin{pmatrix} \text{Jordan–Hölder multiplicity} \\ \text{of } \sigma \text{ in } \mathrm{Sym}^{k-2} \end{pmatrix} \cdot C(\sigma),$$

and the Breuil–Mézard cycles are a $\mathbf{Z}$-basis of the cycles on $\overline{X}_{\overline{r}}^k$.

Each component $Z \subseteq X_{\overline{r}}^k$ has analogous data. Namely, we may consider the number of modular points $N_{\mathrm{aut}}(Z)$ on $Z$, and the cycle expansion

$$[\overline{Z}] = \sum_{\sigma} m_\sigma(\overline{Z}) C(\sigma) \qquad (m_\sigma(\overline{Z}) \in \mathbf{Z}).$$

Here is our answer to Question 1.

**Theorem 1.** *For fixed $k \geq 2$ and any irreducible component $Z \subseteq X_{\overline{r}}^k$,*

$$N_{\mathrm{aut}}(Z) = \sum_{\sigma} m_\sigma(\overline{Z})\, N_{\mathrm{aut}}(\sigma). \tag{1}$$

Theorem 1 is a *local-global distribution* result, since it describes the distribution of the global modular points among the components of the local deformation ring. The left-hand side of (1) is purely global, whereas the right-hand side is mixed. In a sense, the local special fiber geometry plays a larger role than the finite amount of seed data encoded in the $(N_{\mathrm{aut}}(\sigma))_\sigma$, which count modular lifts to Serre weights.

The proof of Theorem 1 builds upon the techniques of Kisin (and its precursors in work of Wiles and Taylor–Wiles and others). We specifically use patching methods to present global deformation rings as formally smooth extensions of local ones, uniformly in the weight $k$, acting on patched modules. What we observe is that the underlying algebra that comes out of the patching method allows one to relate modular point counts to global invariants on the special fibers of the patched deformation rings. In fact, the analysis naturally gives an interesting method also for counting automorphic points on automorphic components for $n$-dimensional Galois representations in terms of global special fiber invariants. The switch from global to local in Theorem 1 is mediated by Breuil–Mézard conjecture, which is only available in limited automorphic contexts (like $\mathrm{GL}_{2/\mathbf{Q}}$).

As presented, Theorem 1 could be interpreted as saying that the number of automorphic points on a component is governed by its special fiber geometry. We also may take the alternate perspective and try to use automorphic methods to control geometry. In this way, we answer Question 2 with the following purely local theorem. Let $I \subseteq D$ be the inertia subgroup and $c_{\mathrm{sm}}(k)$ be the number of *formally smooth* components on $X^k_{\overline{r}}$. Let $\omega$ be the cyclotomic character modulo $p$ and $d_k(\sigma)$ be the Jordan–Hölder multiplicity of $\sigma$ in $\mathrm{Sym}^{k-2}$.

**Theorem 2.** *Assume $p > 7$ and $\overline{r}$ is non-split and reducible such that*

$$\overline{r}|_I \simeq \begin{pmatrix} \omega^{a+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^b$$

*with $2 \leq a \leq p-5$. Then,*

$$c_{\mathrm{sm}}(k) \geq \left(1 - \frac{2}{p} - \frac{4}{p^{\frac{p-3}{2}}} - O(\log k/k)\right) \cdot d_k(\sigma_{a,b}).$$

For $\overline{r}$ as in Theorem 2, $\sigma_{a,b}$ is the unique Serre weight. The cycle $C(\sigma_{a,b})$ is just an irreducible component with multiplicity one. So, it follows from the geometric Breuil–Mézard conjecture that $c_{\mathrm{sm}}(k) \leq d_k(\sigma_{a,b})$. Theorem 2 thus implies the proportion of formally smooth components on $X^k_{\overline{r}}$ is at least roughly $1 - 2/p$. It would be interesting to determine whether the proportion is just 1, as $k \to +\infty$.

The first step in proving Theorem 2 is as follows. Given $Z \subseteq X^k_{\overline{r}}$, to check $Z$ is formally smooth, it suffices to check $m_{\sigma_{a,b}}(\overline{Z}) = 1$. Paškūnas constructed local analogues of patched modules ([10, 3]) using the $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, and applying our local-global theorem to those patched modules, we deduce $Z$ is formally smooth if and only if $Z$ has just one "modular" point coming from the local patched modules.

The assumptions on $\overline{r}$ enter into the second step of the proof, where we use the recent proof of the Bergdall–Pollack *ghost conjecture* ([1, 2]) by Liu, Truong, Xiao, and Zhao ([7, 8]). Those papers prove two remarkable facts for $\overline{r}$ as above. First, each component $Z$ has a well-defined $p$-adic slope, which is a non-negative half-integer. Second, the collection of these slopes (counted with their local modular multiplicities) are exactly the slopes of an elementary *ghost series* Newton polygon that depends just on $\sigma$ and $k$. (The original ghost conjecture did not propose this; the purely local version and its proof is due to Liu, Truong, Xiao, and Zhao.) We conclude that every time a slope appears without multiplicity on the ghost series Newton polygon, it must correspond to a formally smooth component of that slope. The estimate in Theorem 2 is then nothing more than a lower bound for the number of slopes on the ghost series Newton polygon that appear with multiplicity one.

## References


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