S. McDuck
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Duckburg, PA 10101

October 3, 2019

Dear calculus students:

I heard you are about to go on fall break! Congratulations! Given the enterprising spirit of your great TriCo community, I thought now would be the perfect time — when you are less distracted — to hire you for some help.

My money has gone missing!

Many years ago I hired out a company, Parabolic Pit Providers, to build my money pit. Into my pit went my money. I swam there, I relaxed there. It was everything in the world to me!

I have, though, apparently been under attack from underground forces, literally. I spent my time last week ducking around in Boston and at some point during that week my money pit was completely emptied! I had house-sitters keeping an eye on things for me, so I know the thieves have not just lifted the money out. In fact, it looks like the the crime was committed by slowing digging beneath my property to siphon the money out from below. I have drawn you a diagram of the pit just below (I'll explain the dimensions later).

Completely beside myself, I turned to the insurance company. I told them my money was stolen and showed them the giant hole in the bottom of the pit that it disappeared from. They seem to agree I am not a fraud, so they are willing to pay insurance on my gold. The problem is that I left my accounting to my good-for-nothing nephews who can't even manage to put on proper pants. They've lost my accounting book and along with it a record of how much money I actually had!

Luckily, there is a provision in the insurance policy. They will either pay me a lump sum of \$60 billion dollars (that is, 60,000 million) or they will pay me \$2.4 million for each cubic foot of gold. (This is under market, but I'm not in a position to complain). I'm wondering which option to take, here. ***Which option from the insurance company should I take?***

Now you might ask why I am asking *you* this question. Good thought! The thing is I don't know how big my pit is because my blueprints are also missing (nephews...)! I had thought to just fill it with water and measure it that way but the robbers put a hole in the bottom so it cannot hold water anymore. But it just so happens that your professor (Prof. Bergdall at Bryn Mawr College) told me the other day (we have weekly skype meetings) that you all have just learned about using calculus to determine volumes and that your skills are up to this task. I sure hope he's right!

Based on talking it through with Prof. Bergdall, we decided it would helpful to tell you everything we can about the pit:

- We measured its depth. It is 40 feet deep.
- We measured it across. It looks like a circle on top, 40 feet across.
- I also know from the builders that the walls of the pit arc in the shape of a parabola pointing upward.

Is that enough information for you to dig me out of this jam? Let me know if you think not!

Please, please, please, I really need your help! The deadline for me to make my choice with the insurance company is the last Tuesday in October, by end of business (so October 29 at 5pm). At the top of your message back, which I hope you write neatly, can you please just write clearly what your answer to my question is? Also, I do know some calculus but I have forgotten a lot. I know the basics about derivatives and integrals, but not really how to apply them, so I hope your methodology is clear. A representative from the insurance company is going to carefully check your numbers, so make sure all the calculations we need are there also!

Sincerely, with hope and thanks,

Some annotations from your helpful professor I knew ducks could talk, but who knew they could write? (And so well!) Well, I hope you can help old Scrooge out. I though the smallest bit about the problem, and came up with some ideas that might help you dive (!) more deeply into the problem.

- Scrooge's pit sure looks like it's obtained by a revolving process, don't you think?
- It's too bad Scrooge didn't tell us an exact equation for the curving of his walls. Maybe you could start by figuring out a model for the walls.

One more important point. Your helpful professor is traveling the week October 21-25 out of the country. He will be available over email for questions, but you are advised to spend time *before* break thinking about how to pull of the mathematics. Then you can spend your break resting and writing your letter back to Scrooge.

S. Williams
Center court
Flushing Meadow - Corona Park
Flushing, NY 11368

October 31, 2019

Dear calculus students:

Hi, my name is Serena Williams. I am on a break from dominating the world and, as I sit and admire the 23rd of my major open trophies, I wonder: **How likely am I to win a game starting from deuce?**

I've already emailed your always helpful Professor Bergdall. He was in Oaxaca, but he said he knew just the people to help. It's you all! In case you don't know, let me go over how tennis works. In tennis we play rounds, called "games" in tennis lingo, where the same player serves the ball over and over again. In each game, we play for points. A point begins when one person serves. Then we hit the ball back and forth until someone wins the point after the other person makes a mistake. To win a game you have to get to 4 points, *except* you must win by 2. If the score is 3-3, then 4-3 does not win, but 5-3 would. Any time a game is tied at more than 4 points is called *deuce*. **To win from deuce you have to win 2 points in a row.** If we are at deuce and trade two points, one each, we return to deuce and begin again.

Did I mention I'm the most dominant tennis player of all time? I estimate I have a 70% chance of winning any given point. So, to be more precise about my question: **Given I have a 70% chance of winning any point, what is the chance I win starting from deuce**? I was able to work out the chance I win by just winning 2 points right away. On either point, I have a 70% (= 0.7) chance of winning, and so the chance I win 2 points in a row is $0.7 \times 0.7 = 0.49$, for a 49% chance of winning by winning the first 2 points.

The issue is there are more ways to win the game! I can also win by splitting 2 points in a row and then winning 2 points, or by splitting 4 points and then winning 2, and so on. In total it seems like

$$\begin{aligned}\text{Chance I win the game} = \text{Chance I win 1st two points } + \text{ Chance we split 2, then I win 2} + \\ \text{Chance we split 4, then I win 2 } + \text{ Chance we split 6, then I win 2 } + \cdots\end{aligned}$$

The $\cdots$ indicates an infinite sum, and that's why I am reaching out to you! I've never seen infinite anything, but your professor said you've started learning about some infinite kinds of things in calculus. You must be the perfect person to answer my question!

I'd be *so* grateful to get this off of my mind. I took some calculus courses from the internet while I have traveled the world (once again, dominating) but I have not studied this kind of problem. Your answers do not have to be super long, but I hope you can explain in neat writing how you arrived at your answer. Saying right off the bat what chance I have of winning will also help me understand where things are going!

I'll be on the road for the next couple of weeks, but your professor has promised that if you get your response in by Nov 19 at the end of business, then he can first class FedEx it to me. I'd love to read it soon!

Sincerely,

Some annotations from your helpful professor First, the infinite sum business will start to make sense on Halloween but will really come to a head starting on Tuesday November 5. You might want to take a few minutes this weekend to make sure you understand the kind of probability calculation you'll want to make. Serena did a good job of helping you out with the first possible way she can win. Let me know ASAP if you don't get the tennis rules or if you don't understand where Serena got 49%. While we are at it, I'll work out the next case: the chance Serena splits 2 points but then wins by winning 2 points after that.

Let's just say, for concreteness it is you versus Serena and Serena wins 70% of the points, so you win 30%. Then if we look at any 2 straight points we can make a table organizing all the different outcomes of those 2 points:

	Serena wins 1st point	You win 1st point
Serena wins 2nd point	0.49	0.21
You win 2nd point	0.21	0.09

Chances of different outcomes of playing two points in a row versus Serena Williams

So, playing 2 points from deuce Serena has 49% shot of winning, you have a 9% shot of winning and we have a total of 42% chance of splitting the two points and returning to deuce. And so, the overall chance that you split 2 points and then Serena wins the game by winning the next 2 points is going to be

$$0.42 \times 0.49 = 0.2058$$

for an $\approx 21\%$ chance Serena wins the game after exactly 4 points. From here, that should get you started as a model for how to calculate the total chance Serena can win by looking at how Serena can win after 2 points, after 4 points, after 6 points, and so on.

Good luck! Have fun! Office hours and email are always available!

Z. Thomas
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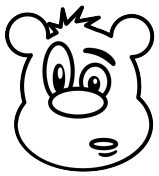
Bryn Mawr, PA 19010



November 19, 2019

Dear calculus students:

Hi, my name is Zadie Thomas. I think my younger sibling Zeke wrote to you earlier this year. You were so much help with his estimation task, we thought you might be able to help me, too. Lucky thing is, mine is way more fun! I like to play the lottery every now and then, even if people say you don't have a real shot, and a few days ago it really paid off. You see, I went to a concert where some musicians were talking about the *harmonic series*. They had a raffle set up where you could win big money. I put my ticket in the "Harmonic Series Patience Test" contest, and I won! My prize is a choice of one of two things:

(i) I can take \$50 cash, immediately, if I want.

(ii) Or, if I wait long enough I will earn \$5,000,000,000,000.

Can you **BELIEVE** that? That is *5 trillion dollars*. More than the US federal spending per year! Given the choices, you gotta feel like there's a catch. So, let me explain how "long" I need to wait and you'll see.

Here is what the contest rules are. Starting when I accept option (ii), a computer is going to start adding up $1 + 1/2 + 1/3 + \cdots$ and so on, **adding one new term to the harmonic series *per second***. After 1 second, the sum will be 1. After 2 seconds, the sum will be $s_2 = 1 + \frac{1}{2} = 1.5$. After three seconds, the sum will be $s_3 = 1 + \frac{1}{2} + \frac{1}{3} \approx 1.833\cdots$. After the n-th second, the sum is going to be $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. **If I can wait long enough to see s_n reach 25, then I will earn the 5 trillion dollars.**

Do you think I should wait for s_n to reach 25?

You might wonder: how do I know the sum will ever reach 25? Well, I texted your always helpful Professor Bergdall and he was able to tell me the "harmonic series" $\sum \frac{1}{n}$ diverges. So, if I wait long enough, the s_n will get as high as I want. For instance, he said that for $n = 50$, you get $s_{50} \approx 4.5$. That's almost 20% to my goal of 25! And after only 50 seconds! This is going to be a breeze!

Then again, the difference between the payoffs in (i) versus (ii) seems quite big. I still have longer to decide what to do, and I'd love for your expertise in figuring out the answer! The contest needs an answer by end of business on Thursday December 5th (5pm). I would so appreciate hearing back from you by then, and be rest assured I will share 10% of my winnings with you! I just hope that makes it more than 5 bucks!

The one thing is that I might be dedicating a lot of time to sitting and waiting for this machine to keep adding things up, so I'd be most grateful if you explained very carefully how long I am going to have to wait for my payday. I have not studied things in calculus involving infinite sums, but I *do* know about integrals (especially like $\int 1/x \, dx = \ln(x) + C$ and stuff like that) so feel free to assume that knowledge.

Sincerely,
Zadie.

Some annotations from your helpful professor This seems like a no-brainer. Right? Not so fast...

(i) You can start by going over Example 3 in Section 9.3 of the textbook one more time. That example explains why the harmonic sum is *a divergent sum*. Look closely and you'll see an estimate

$$1 + \frac{1}{2} + \frac{1}{3} > \ln(4).$$

So that says $s_3 > \ln(4) \approx 1.386$. You could see that by hand, but my hope is that this lower bound for s_3 can be adapted to get a lower bound for s_n. Do you see how the rectangles make that possible?

(ii) On the other hand, we want to know when $s_n \geq 25$ and so we want to know for how many n is $25 > s_n$. To do that, you should find an *upper bound* for s_n using a rectangle idea similar to (i). Looking at the Figure 9.7 (but making some adjustments!) should help. You can also look back at our own slides and discussion about the integral test. Remember, there were two ways we talked about the rectangles!

Good luck! Have fun! Office hours and email are always available!